EVAN CHEN

Euclidean Geometry in Mathematical Olympiads

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Dedicated to the
Mathematical Olympiad Summer Program

Contents

Preface

Give him threepence, since he must make gain out of what he learns.

Euclid of Alexandria

This book is an outgrowth of five years of participating in mathematical olympiads, where geometry flourishes in great vigor. The ideas, techniques, and proofs come from countless resources—lectures at MOP* resources found online, discussions on the Art of Problem Solving site, or even just late-night chats with friends. The problems are taken from contests around the world, many of which I personally solved during the contest, and even a couple of which are my own creations.

As I have learned from these olympiads, mathematical learning is not passive—the only way to learn mathematics is by doing. Hence this book is centered heavily around solving problems, making it especially suitable for students preparing for national or international olympiads. Each chapter contains both examples and practice problems, ranging from easy exercises to true challenges.

Indeed, I was inspired to write this book because as a contestant I did not find any resources I particularly liked. Some books were rich in theory but contained few challenging problems for me to practice on. Other resources I found consisted of hundreds of problems, loosely sorted in topics as broad as "collinearity and concurrence", and lacking any exposition on how a reader should come up with the solutions in the first place. I have thus written this book keeping these issues in mind, and I hope that the structure of the book reflects this.

I am indebted to many people for the materialization of this text. First and foremost, I thank Paul Zeitz for the careful advice he provided that led me to eventually publish this book. I am also deeply indebted to Chris Jeuell and Sam Korsky whose careful readings of the manuscript led to hundreds of revisions and caught errors. Thanks guys!

I also warmly thank the many other individuals who made suggestions and comments on early drafts. In particular, I would like to thank Ray Li, Qing Huang, and Girish Venkat for their substantial contributions, as well as Jingyi Zhao, Cindy Zhang, and Tyler Zhu, among many others. Of course any remaining errors were produced by me and I accept sole responsibility for them. Another special thanks also to the Art of Problem Solving fora,

* The Mathematical Olympiad Summer Program, which is a training program for the USA team at the International Mathematical Olympiad.

from which countless problems in this text were discovered and shared. I would also like to acknowledge Aaron Lin, who I collaborated with on early drafts of the book.

Finally, I of course need to thank everyone who makes the mathematical olympiads possible—the students, the teachers, the problem writers, the coaches, the parents. Math contests not only gave me access to the best peer group in the world but also pushed me to limits that I never could have dreamed were possible. Without them, this book certainly could not have been written.

Evan Chen
Fremont, CA

Preliminaries

0.1 The Structure of This Book

Loosely, each of the chapters is divided into the following parts.

- A theoretical portion, describing a set of related theorems and tools,
- One or more examples demonstrating the application of these tools, and
- A set of several practice problems.

The theoretical portion consists of theorems and techniques, as well as particular geometric configurations. The configurations typically reappear later on, either in the proof of another statement or in the solutions to exercises. Consequently, recognizing a given configuration is often key to solving a particular problem. We present the configurations from the same perspective as many of the problems.

The example problems demonstrate how the techniques in the chapter can be used to solve problems. I have endeavored to not merely provide the solution, but to explain how it comes from, and how a reader would think of it. Often a long commentary precedes the actual formal solution, and almost always this commentary is longer than the solution itself. The hope is to help the reader gain intuition and motivation, which are indispensable for problem solving.

Finally, I have provided roughly a dozen practice problems at the end of each chapter. The hints are numbered and appear in random order in Appendix B, and several of the solutions in Appendix C. I have also tried to include the sources of the problems, so that a diligent reader can find solutions online (for example on the Art of Problem Solving forums, www.aops.com). A full listing of contest acronyms appears in Appendix D.

The book is organized so that earlier chapters never require material from later chapters. However, many of the later chapters approximately commute. In particular, Part III does not rely on Part II. Also, Chapters 6 and 7 can be read in either order. Readers are encouraged to not be bureaucratic in their learning and move around as they see fit, e.g., skipping complicated sections and returning to them later, or moving quickly through familiar material.

0.2 Centers of a Triangle

Throughout the text we refer to several centers of a triangle. For your reference, we define them here.

It is not obvious that these centers exist based on these definitions; we prove this in Chapter 3. For now, you should take their existence for granted.

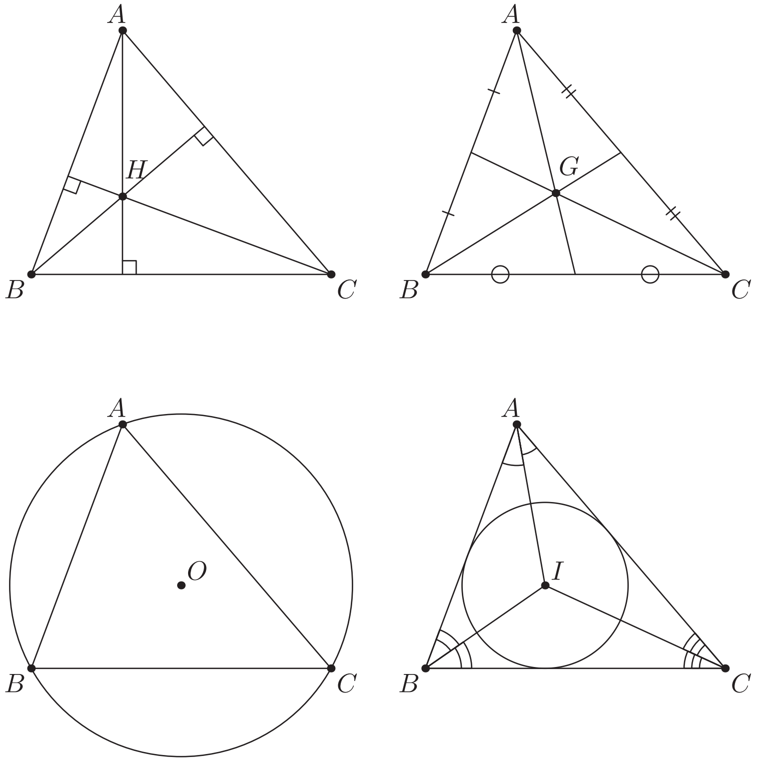

Figure 0.2A. Meet the family! Clockwise from top left: the orthocenter H, centroid G, incenter I, and circumcenter O.

- The **orthocenter** of $\triangle ABC$, usually denoted by H, is the intersection of the perpendiculars (or **altitudes**) from A to $\overline{BC}$, B to $\overline{CA}$, and C to $\overline{AB}$. The triangle formed by the feet of these altitudes is called the **orthic triangle**.
- The **centroid**, usually denoted by G, is the intersection the **medians**, which are the lines joining each vertex to the midpoint of the opposite side. The triangle formed by the midpoints is called the **medial triangle**.
- Next, the **incenter**, usually denoted by I, is the intersection of the angle bisectors of the angles of $\triangle ABC$. It is also the center of a circle (the **incircle**) tangent to all three sides. The radius of the incircle is called the **inradius**.
- Finally, the **circumcenter**, usually denoted by O, is the center of the unique circle (the **circumcircle**) passing through the vertices of $\triangle ABC$. The radius of this circumcircle is called the **circumradius**.

These four centers are shown in Figure 0.2A; we will encounter these remarkable points again and again throughout the book.

0.3 Other Notations and Conventions

Consider a triangle ABC. Throughout this text, let $a = BC$, $b = CA$, $c = AB$, and abbreviate $A = \angle BAC$, $B = \angle CBA$, $C = \angle ACB$ (for example, we may write $\sin \frac{1}{2}A$ for $\sin \frac{1}{2}\angle BAC$). We let

$$s = \frac{1}{2}(a+b+c)$$

denote the **semiperimeter** of $\triangle ABC$.

Next, define $[P_1P_2 \dots P_n]$ to be the area of the polygon $P_1P_2 \dots P_n$. In particular, $[ABC]$ is area of $\triangle ABC$. Finally, given a sequence of points $P_1, P_2, \dots, P_n$ all lying on one circle, let $(P_1P_2 \dots P_n)$ denote this circle.

We use $\measuredangle$ to distinguish a directed angle from a standard angle $\angle$. (Directed angles are defined in Chapter 1.) Angles are measured in degrees.

Finally, we often use the notation $\overline{AB}$ to denote either the segment AB or the line AB; the use should be clear from context. In the rare case we need to make a distinction we explicitly write out "line AB" or "segment AB". Beginning in Chapter 9, we also use the shorthand $\overline{AB} \cap \overline{CD}$ for the intersection of the two lines $\overline{AB}$ and $\overline{CD}$.

In long algebraic computations which have some amount of symmetry, we may use **cyclic sum notation** as follows: the notation

$$\sum_{\text{cyc}} f(a,b,c)$$

is shorthand for the cyclic sum

$$f(a,b,c) + f(b,c,a) + f(c,a,b).$$

For example,

$$\sum_{\text{cyc}} a^2b = a^2b + b^2c + c^2a.$$

Part I
Fundamentals

CHAPTER 1

Angle Chasing

> This is your last chance. After this, there is no turning back. You take the blue pill—the story ends, you wake up in your bed and believe whatever you want to believe. You take the red pill—you stay in Wonderland and I show you how deep the rabbit-hole goes.
>
> Morpheus in *The Matrix*

Angle chasing is one of the most fundamental skills in olympiad geometry. For that reason, we dedicate the entire first chapter to fully developing the technique.

1.1 Triangles and Circles

Consider the following example problem, illustrated in Figure 1.1A.

Example 1.1. In quadrilateral $WXYZ$ with perpendicular diagonals (as in Figure 1.1A), we are given $\angle WZX = 30^\circ$, $\angle XWY = 40^\circ$, and $\angle WYZ = 50^\circ$.

(a) Compute $\angle Z$.
(b) Compute $\angle X$.

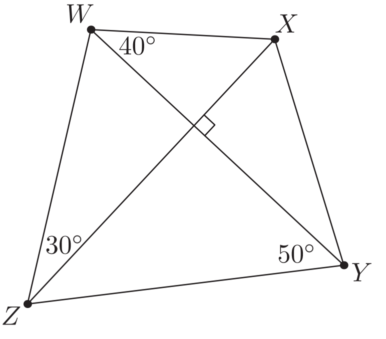

Figure 1.1A. Given these angles, which other angles can you compute?

You probably already know the following fact:

Proposition 1.2 (Triangle Sum). *The sum of the angles in a triangle is* 180°.

As it turns out, this is not sufficient to solve the entire problem, only the first half. The next section develops the tools necessary for the second half. Nevertheless, it is perhaps surprising what results we can derive from Proposition 1.2 alone. Here is one of the more surprising theorems.

Theorem 1.3 (Inscribed Angle Theorem). *If $\angle ACB$ is inscribed in a circle, then it subtends an arc with measure $2\angle ACB$.*

Proof. Draw in $\overline{OC}$. Set $\alpha = \angle ACO$ and $\beta = \angle BCO$, and let $\theta = \alpha + \beta$.

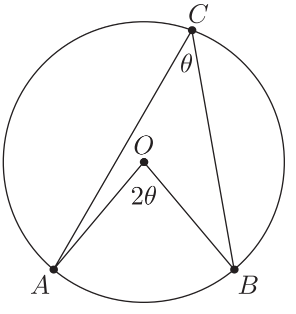

Figure 1.1B. The inscribed angle theorem.

We need some way to use the condition $AO = BO = CO$. How do we do so? Using isosceles triangles, roughly the only way we know how to convert lengths into angles. Because $AO = CO$, we know that $\angle OAC = \angle OCA = \alpha$. How does this help? Using Proposition 1.2 gives

$$\angle AOC = 180^\circ - (\angle OAC + \angle OCA) = 180^\circ - 2\alpha.$$

Now we do exactly the same thing with B. We can derive

$$\angle BOC = 180^\circ - 2\beta.$$

Therefore,

$$\angle AOB = 360^\circ - (\angle AOC + \angle BOC) = 360^\circ - (360^\circ - 2\alpha - 2\beta) = 2\theta$$

and we are done. □

We can also get information about the centers defined in Section 0.2. For example, recall the *incenter* is the intersection of the angle bisectors.

Example 1.4. If I is the incenter of $\triangle ABC$ then

$$\angle BIC = 90^\circ + \frac{1}{2}A.$$

Proof. We have

$$\begin{aligned}\angle BIC &= 180^\circ - (\angle IBC + \angle ICB)\\ &= 180^\circ - \frac{1}{2}(B+C)\\ &= 180^\circ - \frac{1}{2}(180^\circ - A)\\ &= 90^\circ + \frac{1}{2}A.\end{aligned}$$ □

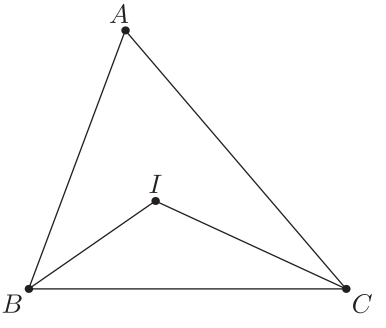

Figure 1.1C. The incenter of a triangle.

Problems for this Section

Problem 1.5. Solve the first part of Example 1.1. **Hint**: 185

Problem 1.6. Let ABC be a triangle inscribed in a circle ω. Show that $\overline{AC} \perp \overline{CB}$ if and only if $\overline{AB}$ is a diameter of ω.

Problem 1.7. Let O and H denote the circumcenter and orthocenter of an acute $\triangle ABC$, respectively, as in Figure 1.1D. Show that $\angle BAH = \angle CAO$. **Hints**: 540 373

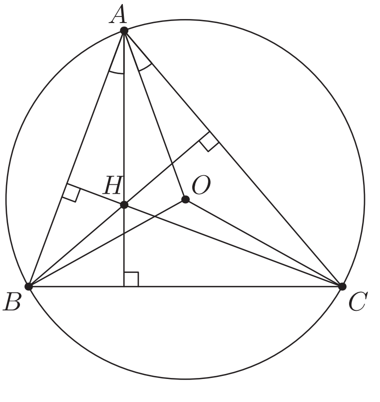

Figure 1.1D. The orthocenter and circumcenter. See Section 0.2 if you are not familiar with these.

1.2 Cyclic Quadrilaterals

The heart of this section is the following proposition, which follows directly from the inscribed angle theorem.

Proposition 1.8. *Let $ABCD$ be a convex cyclic quadrilateral. Then $\angle ABC + \angle CDA = 180^\circ$ and $\angle ABD = \angle ACD$.*

Here a **cyclic quadrilateral** refers to a quadrilateral that can be inscribed in a circle. See Figure 1.2A. More generally, points are **concyclic** if they all lie on some circle.

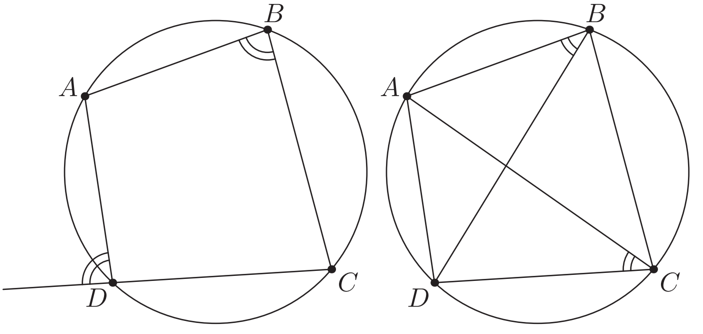

Figure 1.2A. Cyclic quadrilaterals with angles marked.

At first, this result seems not very impressive in comparison to our original theorem. However, it turns out that the converse of the above fact is true as well. Here it is more explicitly.

Theorem 1.9 (Cyclic Quadrilaterals). *Let $ABCD$ be a convex quadrilateral. Then the following are equivalent:*

(i) $ABCD$ *is cyclic.*
(ii) $\angle ABC + \angle CDA = 180^\circ$.
(iii) $\angle ABD = \angle ACD$.

This turns out to be extremely useful, and several applications appear in the subsequent sections. For now, however, let us resolve the problem we proposed at the beginning.

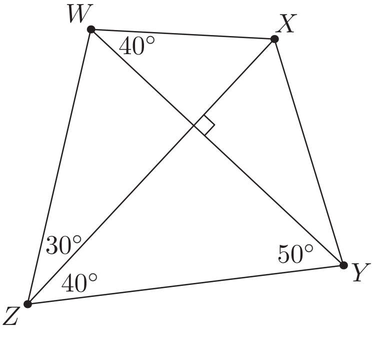

Figure 1.2B. Finishing Example 1.1. We discover $WXYZ$ is cyclic.

Solution to Example 1.1, part (b). Let P be the intersection of the diagonals. Then we have $\angle PZY = 90^\circ - \angle PYZ = 40^\circ$. Add this to the figure to obtain Figure 1.2B.

Now consider the 40° angles. They satisfy condition (iii) of Theorem 1.9. That means the quadrilateral $WXYZ$ is cyclic. Then by condition (ii), we know

$$\angle X = 180^\circ - \angle Z$$

Yet $\angle Z = 30^\circ + 40^\circ = 70^\circ$, so $\angle X = 110^\circ$, as desired. □

In some ways, this solution is totally unexpected. Nowhere in the problem did the problem mention a circle; nowhere in the solution does its center ever appear. And yet, using the notion of a cyclic quadrilateral reduced it to a mere calculation, whereas the problem was not tractable beforehand. This is where Theorem 1.9 draws its power.

We stress the importance of Theorem 1.9. It is not an exaggeration to say that more than 50% of standard olympiad geometry problems use it as an intermediate step. We will see countless applications of this theorem throughout the text.

Problems for this Section

Problem 1.10. Show that a trapezoid is cyclic if and only if it is isosceles.

Problem 1.11. Quadrilateral $ABCD$ has $\angle ABC = \angle ADC = 90^\circ$. Show that $ABCD$ is cyclic, and that $(ABCD)$ (that is, the circumcircle of $ABCD$) has diameter $\overline{AC}$.

1.3 The Orthic Triangle

In $\triangle ABC$, let D, E, F denote the feet of the altitudes from A, B, and C. The $\triangle DEF$ is called the **orthic triangle** of $\triangle ABC$. This is illustrated in Figure 1.3A.

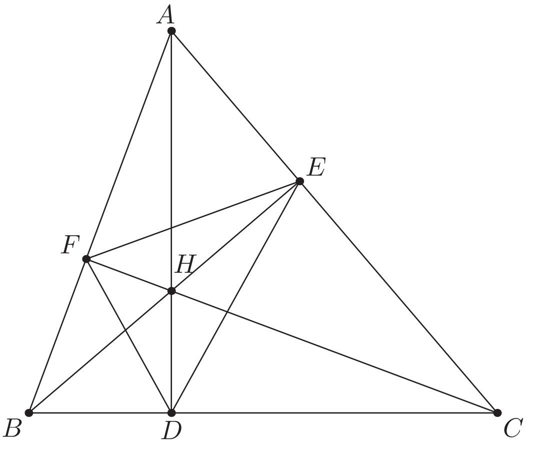

Figure 1.3A. The orthic triangle.

It also turns out that lines AD, BE, and CF all pass through a common point H, which is called the **orthocenter** of H. We will show the orthocenter exists in Chapter 3.

Although there are no circles drawn in the figure, the diagram actually contains six cyclic quadrilaterals.

Problem 1.12. In Figure 1.3A, there are six cyclic quadrilaterals with vertices in $\{A, B, C, D, E, F, H\}$. What are they? **Hint**: 91

To get you started, one of them is $AFHE$. This is because $\angle AFH = \angle AEH = 90^\circ$, and so we can apply (ii) of Theorem 1.9. Now find the other five!

Once the quadrilaterals are found, we are in a position of power; we can apply any part of Theorem 1.9 freely to these six quadrilaterals. (In fact, you can say even more—the right angles also tell you where the diameter of the circle is. See Problem 1.6.) Upon closer inspection, one stumbles upon the following.

Example 1.13. Prove that H is the incenter of $\triangle DEF$.

Check that this looks reasonable in Figure 1.3A.

We encourage the reader to try this problem before reading the solution below.

Solution to Example 1.13. Refer to Figure 1.3A. We prove that $\overline{DH}$ is the bisector of $\angle EDF$. The other cases are identical, and left as an exercise.

Because $\angle BFH = \angle BDH = 90^\circ$, we see that $BFHD$ is cyclic by Theorem 1.9. Applying the last clause of Theorem 1.9 again, we find

$$\angle FDH = \angle FBH.$$

Similarly, $\angle HEC = \angle HDC = 90^\circ$, so $CEHD$ is cyclic. Therefore,

$$\angle HDE = \angle HCE.$$

Because we want to prove that $\angle FDH = \angle HDE$, we only need to prove that $\angle FBH = \angle HCE$; in other words, $\angle FBE = \angle FCE$. This is equivalent to showing that $FBCE$ is cyclic, which follows from $\angle BFC = \angle BEC = 90^\circ$. (One can also simply show that both are equal to $90^\circ - A$ by considering right triangles BEA and CFA.)

Hence, $\overline{DH}$ is indeed the bisector, and therefore we conclude that H is the incenter of $\triangle DEF$. $\square$

Combining the results of the above, we obtain our first configuration.

Lemma 1.14 (The Orthic Triangle). *Suppose $\triangle DEF$ is the orthic triangle of acute $\triangle ABC$ with orthocenter H. Then*

(a) Points A, E, F, H lie on a circle with diameter $\overline{AH}$.
(b) Points B, E, F, C lie on a circle with diameter $\overline{BC}$.
(c) H is the incenter of $\triangle DEF$.

Problems for this Section

Problem 1.15. Work out the similar cases in the solution to Example 1.13. That is, explicitly check that $\overline{EH}$ and $\overline{FH}$ are actually bisectors as well.

Problem 1.16. In Figure 1.3A, show that $\triangle AEF$, $\triangle BFD$, and $\triangle CDE$ are each similar to $\triangle ABC$. **Hint**: 181

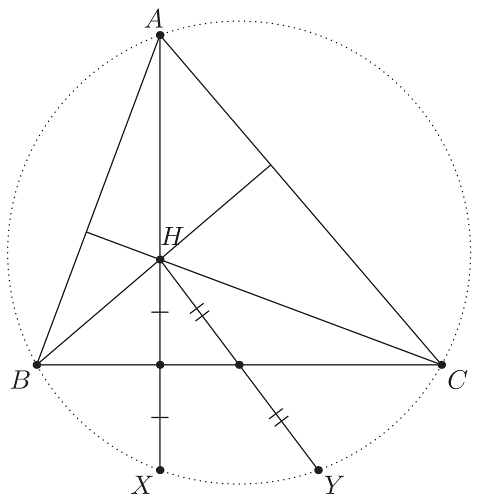

Figure 1.3B. Reflecting the orthocenter. See Lemma 1.17.

Lemma 1.17 (Reflecting the Orthocenter). *Let H be the orthocenter of $\triangle ABC$, as in Figure 1.3B. Let X be the reflection of H over $\overline{BC}$ and Y the reflection over the midpoint of $\overline{BC}$.*

(a) Show that X lies on (ABC).
(b) Show that $\overline{AY}$ is a diameter of (ABC). **Hint**: 674

1.4 The Incenter/Excenter Lemma

We now turn our attention from the orthocenter to the incenter. Unlike before, the cyclic quadrilateral is essentially given to us. We can use it to produce some interesting results.

Lemma 1.18 (The Incenter/Excenter Lemma). *Let ABC be a triangle with incenter I. Ray AI meets (ABC) again at L. Let I_A be the reflection of I over L. Then,*

(a) The points I, B, C, and I_A lie on a circle with diameter $\overline{II_A}$ and center L. In particular, $LI = LB = LC = LI_A$.
(b) Rays BI_A and CI_A bisect the exterior angles of $\triangle ABC$.

By "exterior angle", we mean that ray BI_A bisects the angle formed by the segment BC and the extension of line AB past B. The point I_A is called the **A-excenter*** of $\triangle ABC$; we visit it again in Section 2.6.

Let us see what we can do with cyclic quadrilateral $ABLC$.

* Usually the A-excenter is defined as the intersection of exterior angle bisectors of $\angle B$ and $\angle C$, rather than as the reflection of I over L. In any case, Lemma 1.18 shows these definitions are equivalent.

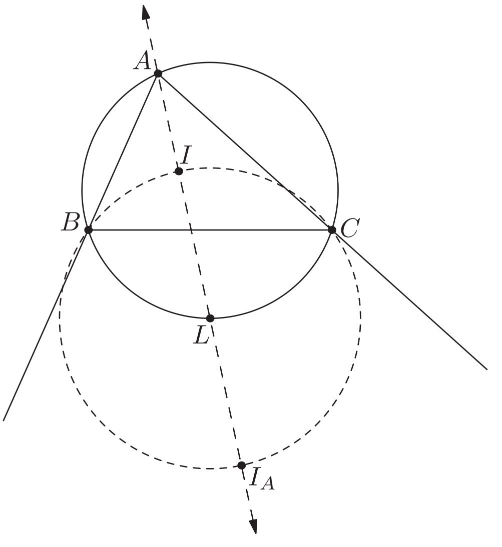

Figure 1.4A. Lemma 1.18, the incenter/excenter lemma.

Proof. Let $\angle A = 2\alpha$, $\angle B = 2\beta$, and $\angle C = 2\gamma$ and notice that $\angle A + \angle B + \angle C = 180^\circ \Rightarrow \alpha + \beta + \gamma = 90^\circ$.

Our first goal is to prove that $LI = LB$. We prove this by establishing $\angle IBL = \angle LIB$ (this lets us convert the conclusion completely into the language of angles). To do this, we invoke (iii) of Theorem 1.9 to get $\angle CBL = \angle LAC = \angle IAC = \alpha$. Therefore,

$$\angle IBL = \angle IBC + \angle CBL = \beta + \alpha.$$

All that remains is to compute $\angle BIL$. But this is simple, as

$$\angle BIL = 180^\circ - \angle AIB = \angle IBA + \angle BAI = \alpha + \beta$$

Therefore triangle LBI is isosceles, with $LI = LB$, which is what we wanted.

Similar calculations give $LI = LC$.

Because $LB = LI = LC$, we see that L is indeed the center of (IBC). Because L is given to be the midpoint of $\overline{II_A}$, it follows that $\overline{II_A}$ is a diameter of (LBC) as well.

Let us now approach the second part. We wish to show that $\angle I_A BC = \frac{1}{2}(180^\circ - 2\beta) = 90^\circ - \beta$. Recalling that $\overline{II_A}$ is a diameter of the circle, we observe that

$$\angle IBI_A = \angle ICI_A = 90^\circ.$$

so $\angle I_A BC = \angle I_A BI - \angle IBC = 90^\circ - \beta$.

Similar calculations yield that $\angle BCI_A = 90^\circ - \gamma$, as required. $\square$

This configuration shows up very often in olympiad geometry, so recognize it when it appears!

Problem for this Section

Problem 1.19. Fill in the two similar calculations in the proof of Lemma 1.18.

1.5 Directed Angles

Some motivation is in order. Look again at Figure 1.3A. We assumed that $\triangle ABC$ was acute. What happens if that is not true? For example, what if $\angle A > 90^\circ$ as in Figure 1.5A?

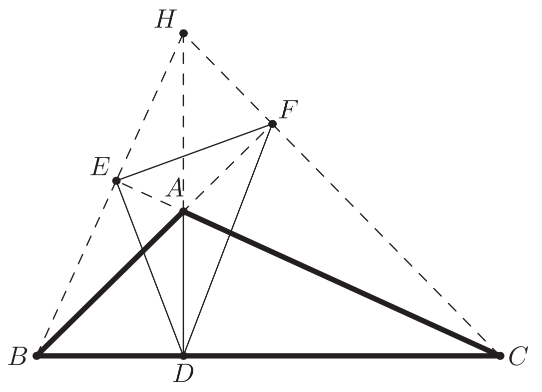

Figure 1.5A. No one likes configuration issues.

There should be something scary in the above figure. Earlier, we proved that points B, E, A, D were concyclic using criterion (iii) of Theorem 1.9. Now, the situation is different. Has anything changed?

Problem 1.20. Recall the six cyclic quadrilaterals from Problem 1.12. Check that they are still cyclic in Figure 1.5A.

Problem 1.21. Prove that, in fact, A is the orthocenter of $\triangle HBC$.

In this case, we are okay, but the dangers are clear. For example, when $\triangle ABC$ was acute, we proved that B, H, F, D were concyclic by noticing that the opposite angles satisfied $\angle BDH + \angle HFB = 180^\circ$. Here, however, we instead have to use the fact that $\angle BDH = \angle BFH$; in other words, for the same problem we have to use different parts of Theorem 1.9. We should not need to worry about solving the same problem twice!

How do we handle this? The solution is to use **directed angles** mod 180°. Such angles will be denoted with a $\measuredangle$ symbol instead of the standard $\angle$. (This notation is not standard; should you use it on a contest, do not neglect to say so in the opening lines of your solution.)

Here is how it works. First, we consider $\measuredangle ABC$ to be *positive* if the vertices A, B, C appear in clockwise order, and *negative* otherwise. In particular, $\measuredangle ABC \neq \measuredangle CBA$; they are negatives. See Figure 1.5B.

Then, we are taking the angles modulo 180°. For example,

$$-150^\circ = 30^\circ = 210^\circ.$$

Why on earth would we adopt such a strange convention? The key is that our Theorem 1.9 can now be rewritten as follows.

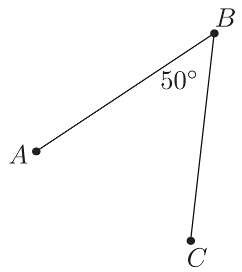

Figure 1.5B. Here, $\measuredangle ABC = 50^\circ$ and $\measuredangle CBA = -50^\circ$.

Theorem 1.22 (Cyclic Quadrilaterals with Directed Angles). *Points A, B, X, Y lie on a circle if and only if*

$$\measuredangle AXB = \measuredangle AYB.$$

This seems too good to be true, as we have dropped the convex condition—there is now only one case of the theorem. In other words, as long as we direct our angles, we no longer have to worry about configuration issues when applying Theorem 1.9.

Problem 1.23. Verify that parts (ii) and (iii) of Theorem 1.9 match the description in Theorem 1.22.

We present some more convenient truths in the following proposition.

Proposition 1.24 (Directed Angles). *For any distinct points A, B, C, P in the plane, we have the following rules.*

Oblivion. $\measuredangle APA = 0$.

Anti-Reflexivity. $\measuredangle ABC = -\measuredangle CBA$.

Replacement. $\measuredangle PBA = \measuredangle PBC$ *if and only if A, B, C are collinear. (What happens when $P = A$?) Equivalently, if C lies on line BA, then the A in $\measuredangle PBA$ may be replaced by C.*

Right Angles. *If $\overline{AP} \perp \overline{BP}$, then $\measuredangle APB = \measuredangle BPA = 90^\circ$.*

Directed Angle Addition. $\measuredangle APB + \measuredangle BPC = \measuredangle APC$.

Triangle Sum. $\measuredangle ABC + \measuredangle BCA + \measuredangle CAB = 0$.

Isosceles Triangles. $AB = AC$ *if and only if* $\measuredangle ACB = \measuredangle CBA$.

Inscribed Angle Theorem. *If (ABC) has center P, then $\measuredangle APB = 2\measuredangle ACB$.*

Parallel Lines. *If $\overline{AB} \parallel \overline{CD}$, then $\measuredangle ABC + \measuredangle BCD = 0$.*

One thing we have to be careful about is that $2\measuredangle ABC = 2\measuredangle XYZ$ does not imply $\measuredangle ABC = \measuredangle XYZ$, because we are taking angles modulo 180°. Hence it does not make sense to take half of a directed angle.[†]

Problem 1.25. Convince yourself that all the claims in Proposition 1.24 are correct.

[†] Because of this, it is customary to take arc measures modulo 360°. We may then write the inscribed angle theorem as $\measuredangle ABC = \frac{1}{2}\widehat{AC}$. This is okay since $\measuredangle ABC$ is taken mod 180° but $\widehat{AC}$ is taken mod 360°.

Directed angles are quite counterintuitive at first, but with a little practice they become much more natural. The right way to think about them is to solve the problem for a specific configuration, but write down all statements in terms of directed angles. The solution for a specific configuration then automatically applies to all configurations.

Before moving in to a less trivial example, let us finish the issue with the orthic triangle once and for all.

Example 1.26. Let H be the orthocenter of $\triangle ABC$, acute or not. Using directed angles, show that $AEHF$, $BFHD$, $CDHE$, $BEFC$, $CFDA$, and $ADEB$ are cyclic.

Solution. We know that

$$90^\circ = \measuredangle ADB = \measuredangle ADC$$
$$90^\circ = \measuredangle BEC = \measuredangle BEA$$
$$90^\circ = \measuredangle CFA = \measuredangle CFB$$

because of right angles. Then

$$\measuredangle AEH = \measuredangle AEB = -\measuredangle BEA = -90^\circ = 90^\circ$$

and

$$\measuredangle AFH = \measuredangle AFC = -\measuredangle CFA = -90^\circ = 90^\circ$$

so A, E, F, H are concyclic. Also,

$$\measuredangle BFC = -\measuredangle CFB = -90^\circ = 90^\circ = \measuredangle BEC$$

so B, E, F, C are concyclic. The other quadrilaterals have similar stories. □

We conclude with one final example.

Lemma 1.27 (Miquel Point of a Triangle). *Points D, E, F lie on lines BC, CA, and AB of $\triangle ABC$, respectively. Then there exists a point lying on all three circles (AEF), (BFD), (CDE).*

This point is often called the **Miquel point** of the triangle.

It should be clear by looking at Figure 1.5C that many, many configurations are possible. Trying to handle this with standard angles would be quite messy. Fortunately, we can get them all in one go with directed angles.

Let K be the intersection of (BFD) and (CDE) other than D. The goal is to show that $AFEK$ is cyclic as well. For the case when K is inside $\triangle ABC$, this is an easy angle chase. All we need to do is use the corresponding statements with directed angles for each step.

We strongly encourage readers to try this themselves before reading the solution that follows.

First, here is the solution for the first configuration of Figure 1.5C. Define K as above. Now we just notice that $\angle FKD = 180^\circ - B$ and $\angle EKD = 180^\circ - C$. Consequently, $\angle FKE = 360^\circ - (180^\circ - C) - (180^\circ - B) = B + C = 180^\circ - A$ and $AFEK$ is cyclic. Now we just need to convert this into directed angles.

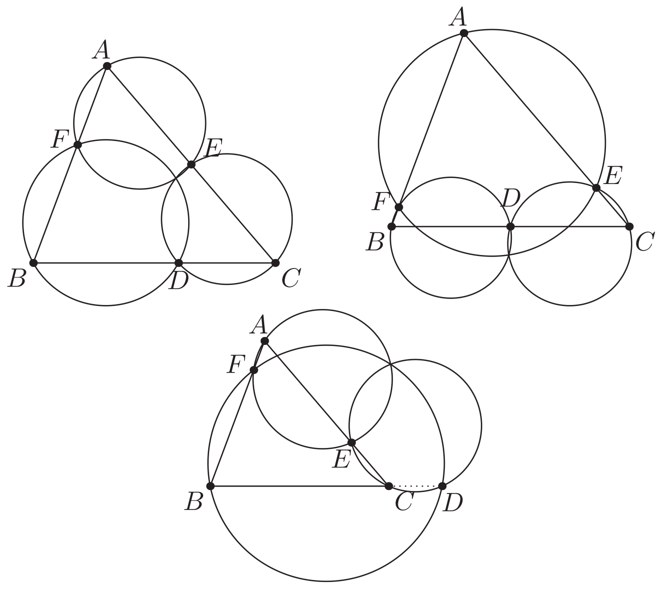

Figure 1.5C. The Miquel point, as in Lemma 1.27.

Proof. The first two claims are just

$$\measuredangle FKD = \measuredangle FBD = \measuredangle ABC \text{ and } \measuredangle DKE = \measuredangle DCE = \measuredangle BCA.$$

We also know that

$$\measuredangle FKD + \measuredangle DKE + \measuredangle EKF = 0 \text{ and } \measuredangle ABC + \measuredangle BCA + \measuredangle CAB = 0.$$

The first equation represents the fact that the sum of the angles at K is $360°$; the second is the fact that the sum of the angles in a triangle is $180°$. From here we derive that $\measuredangle CAB = \measuredangle EKF$. But $\measuredangle CAB = \measuredangle EAF$; hence $\measuredangle EAF = \measuredangle EKF$ as desired. □

Having hopefully convinced you that directed angles are natural and often useful, let us provide a warning on when not to use them. Most importantly, you should not use directed angles when the problem only works for a certain configuration! An example of this is Problem 1.38; the problem statement becomes false if the quadrilateral is instead $ABDC$. You should also avoid using directed angles if you need to invoke trigonometry, or if you need to take half an angle (as in Problem 1.38 again). These operations do not make sense modulo $180°$.

Problems for this Section

Problem 1.28. We claimed that $\measuredangle FKD + \measuredangle DKE + \measuredangle EKF = 0$ in the above proof. Verify this using Proposition 1.24.

Problem 1.29. Show that for any distinct points A, B, C, D we have $\measuredangle ABC + \measuredangle BCD + \measuredangle CDA + \measuredangle DAB = 0$. **Hints**: 114 645

Lemma 1.30. *Points A, B, C lie on a circle with center O. Show that $\measuredangle OAC = 90^\circ - \measuredangle CBA$. (This is not completely trivial.)* **Hints**: 8 530 109

1.6 Tangents to Circles and Phantom Points

Here we introduce one final configuration and one general technique.

First, we discuss the **tangents** to a circle. In many ways, one can think of it as Theorem 1.22 applied to the "quadrilateral" $AABC$. Indeed, consider a point X on the circle and the line XA. As we move X closer to A, the line XA approaches the tangent at A. The limiting case becomes the theorem below.

Proposition 1.31 (Tangent Criterion). *Suppose $\triangle ABC$ is inscribed in a circle with center O. Let P be a point in the plane. Then the following are equivalent:*

(i) $\overline{PA}$ *is tangent to* (ABC).
(ii) $\overline{OA} \perp \overline{AP}$.
(iii) $\measuredangle PAB = \measuredangle ACB$.

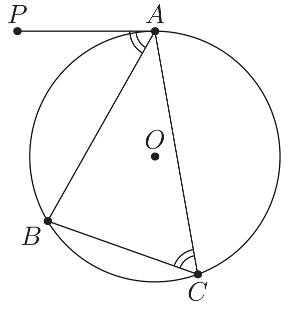

Figure 1.6A. PA is a tangent to (ABC). See Proposition 1.31.

In the following example we also introduce the technique of adding a **phantom point**. (This general theme is sometimes also called **reverse reconstruction**.)

Example 1.32. Let ABC be an acute triangle with circumcenter O, and let K be a point such that $\overline{KA}$ is tangent to (ABC) and $\angle KCB = 90^\circ$. Point D lies on $\overline{BC}$ such that $\overline{KD} \parallel \overline{AB}$. Show that line $\overline{DO}$ passes through A.

This problem is perhaps a bit trickier to solve directly, because we have not developed any tools to show that three points are collinear. (We will!) But here is a different idea. We define a phantom point D' as the intersection of ray AO with $\overline{BC}$. If we can show that $\overline{KD'} \parallel \overline{AB}$, then this will prove $D' = D$, because there is only one point on $\overline{BC}$ with $\overline{KD} \parallel \overline{AB}$.

Fortunately, this can be done with merely the angle chasing that we know earlier. We leave it as Problem 1.33. As a hint, you will have to use both parts of Proposition 1.31.

We have actually encountered a similar idea before, in our proof of Lemma 1.27. The idea was to let (BDF) and (CDE) intersect at a point K, and then show that K was on the

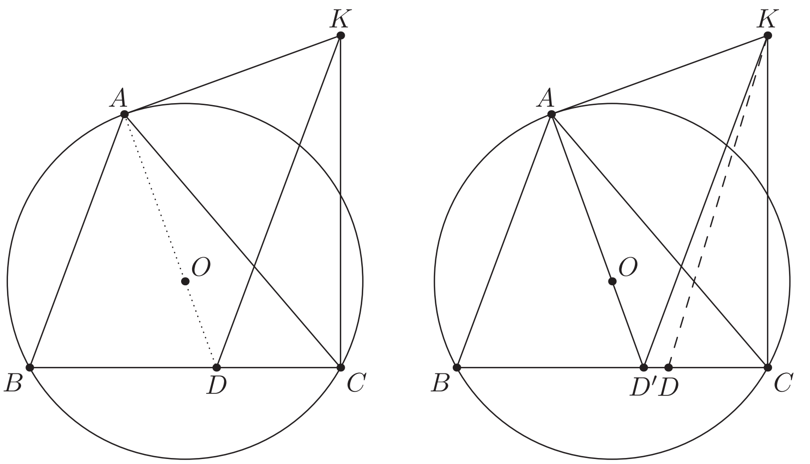

Figure 1.6B. Example 1.32, and the phantom point.

third circle as well. This theme is common in geometry. A second example where phantom points are helpful is Lemma 1.45 on page 19.

It is worth noting that solutions using phantom points can often (but not always) be rearranged to avoid them, although such solutions may be much less natural. For example, another way to solve Example 1.32 is to show that $\measuredangle KAO = \measuredangle KAD$. Problem 1.34 is the most common example of a problem that is not easy to rewrite without phantom points.

Problems for this Section

Problem 1.33. Let ABC be a triangle and let ray AO meet $\overline{BC}$ at D'. Point K is selected so that $\overline{KA}$ is tangent to (ABC) and $\angle KC = 90^\circ$. Prove that $\overline{KD'} \parallel \overline{AB}$.

Problem 1.34. In scalene triangle ABC, let K be the intersection of the angle bisector of $\angle A$ and the perpendicular bisector of $\overline{BC}$. Prove that the points A, B, C, K are concyclic.
Hints: 356 101

1.7 Solving a Problem from the IMO Shortlist

To conclude the chapter, we leave the reader with one last example problem. We hope the discussion is instructive.

Example 1.35 (Shortlist 2010/G1). Let ABC be an acute triangle with D, E, F the feet of the altitudes lying on $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ respectively. One of the intersection points of the line EF and the circumcircle is P. The lines BP and DF meet at point Q. Prove that $AP = AQ$.

In this problem there are two possible configurations. Directed angles allows us to handle both, but let us focus on just one—say P_2 and Q_2.

The first thing we notice is the orthic triangle. Because of it we should keep the results of Lemma 1.14 close at heart. Additionally, we are essentially given that $ACBP_2$ is a cyclic

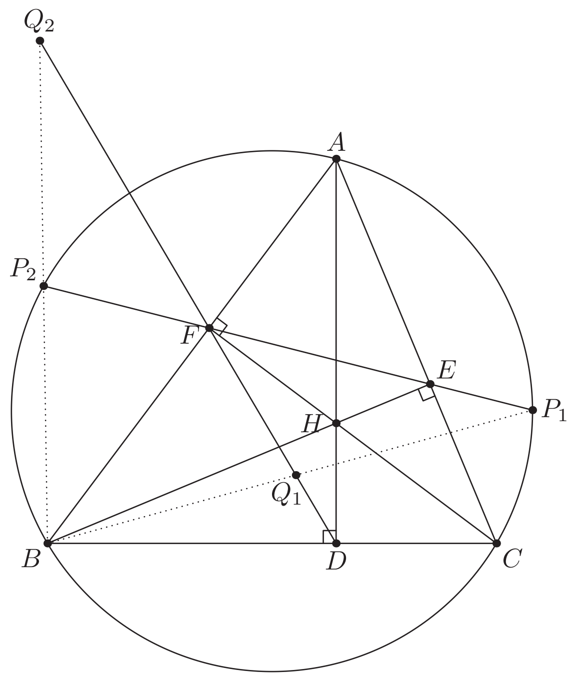

Figure 1.7A. IMO Shortlist 2010, Problem G1 (Example 1.35).

quadrilateral. Let us see what we can do with that. The conclusion $AP_2 = AQ_2$ seems better expressed in terms of angles—we want to show that $\measuredangle AQ_2P_2 = \measuredangle Q_2P_2A$. Now we already know $\measuredangle Q_2P_2A$, because

$$\measuredangle Q_2P_2A = \measuredangle BP_2A = \measuredangle BCA$$

so it is equivalent to compute $\measuredangle AQ_2P_2$.

There are two ways to realize the next step. The first is wishful thinking—the hope that a convenient cyclic quadrilateral will give us $\measuredangle AQ_2P_2$. The second way is to have a scaled diagram at hand. Either way, we stumble upon the following hope: might AQ_2P_2F be cyclic? It certainly looks like it in the diagram.

How might we prove that AQ_2P_2F is cyclic? Trying to use supplementary angles seems not as hopeful, because this is what we want to use as a final step. However, inscribed arcs seems more promising. We already know $\measuredangle AP_2Q_2 = \measuredangle ACB$. Might we be able to find AFQ_2? Yes—we know that

$$\measuredangle AFQ_2 = \measuredangle AFD$$

and now we are certain this will succeed, because $\measuredangle AFD$ is entirely within the realm of $\triangle ABC$ and its orthic triangle. In other words, we have eliminated P and Q. In fact,

$$\measuredangle AFD = \measuredangle ACD = \measuredangle ACB$$

since $AFDC$ is cyclic. This solves the problem for P_2 and Q_2. Because we have been careful to direct all the angles, this automatically solves the case P_1 and Q_1 as well—and this is why directed angles are useful.

It is important to realize that the above is not a well-written proof, but instead a description of how to arrive at the solution. Below is an example of how to write the proof in a contest—one direction only (so without working backwards like we did at first), and without the motivation. Follow along in the following proof with P_1 and Q_1, checking that the directed angles work out.

Solution to Example 1.35. First, because $APBC$ and $AFDC$ are cyclic,

$$\measuredangle QPA = \measuredangle BPA = \measuredangle BCA = \measuredangle DCA = \measuredangle DFA = \measuredangle QFA.$$

Therefore, we see $AFPQ$ is cyclic. Then

$$\measuredangle AQP = \measuredangle AFP = \measuredangle AFE = \measuredangle AHE = \measuredangle DHE = \measuredangle DCE = \measuredangle BCA.$$

We deduce that $\measuredangle AQP = \measuredangle BCA = \measuredangle QPA$ which is enough to imply that $\triangle APQ$ is isosceles with $AP = AQ$. □

This problem is much easier if Lemma 1.14 is kept in mind. In that case, the only key observation is that $AFPQ$ is cyclic. As we saw above, one way to make this key observation is to merely peruse the diagram for quadrilaterals that appear cyclic. That is why it is often a good idea, on any contest problem, to draw a scaled diagram using ruler and compass—in fact, preferably more than one diagram. This often gives away intermediate steps in the problem, prevents you from missing obvious facts, or gives you something to attempt to prove. It will also prevent you from wasting time trying to prove false statements.

1.8 Problems

Problem 1.36. Let $ABCDE$ be a convex pentagon such that $BCDE$ is a square with center O and $\angle A = 90°$. Prove that $\overline{AO}$ bisects $\angle BAE$. **Hints**: 18 115 **Sol:** p.241

Problem 1.37 (BAMO 1999/2). Let $O = (0, 0)$, $A = (0, a)$, and $B = (0, b)$, where $0 < a < b$ are reals. Let Γ be a circle with diameter $\overline{AB}$ and let P be any other point on Γ. Line PA meets the x-axis again at Q. Prove that $\angle BQP = \angle BOP$. **Hints**: 635 100

Problem 1.38. In cyclic quadrilateral $ABCD$, let I_1 and I_2 denote the incenters of $\triangle ABC$ and $\triangle DBC$, respectively. Prove that I_1I_2BC is cyclic. **Hints**: 684 569

Problem 1.39 (CGMO 2012/5). Let ABC be a triangle. The incircle of $\triangle ABC$ is tangent to $\overline{AB}$ and $\overline{AC}$ at D and E respectively. Let O denote the circumcenter of $\triangle BCI$.

Prove that $\angle ODB = \angle OEC$. **Hints**: 643 89 **Sol:** p.241

Problem 1.40 (Canada 1991/3). Let P be a point inside circle ω. Consider the set of chords of ω that contain P. Prove that their midpoints all lie on a circle. **Hints**: 455 186 169

Problem 1.41 (Russian Olympiad 1996). Points E and F are on side $\overline{BC}$ of convex quadrilateral $ABCD$ (with E closer than F to B). It is known that $\angle BAE = \angle CDF$ and $\angle EAF = \angle FDE$. Prove that $\angle FAC = \angle EDB$. **Hints**: 245 614

Lemma 1.42. *Let ABC be an acute triangle inscribed in circle Ω. Let X be the midpoint of the arc $\widehat{BC}$ not containing A and define Y, Z similarly. Show that the orthocenter of XYZ is the incenter I of ABC.* **Hints**: 432 21 326 195

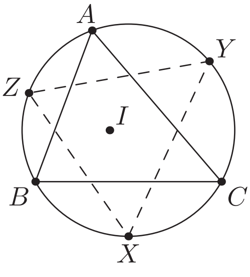

Figure 1.8A. Lemma 1.42. I is the orthocenter of $\triangle XYZ$.

Problem 1.43 (JMO 2011/5). Points A, B, C, D, E lie on a circle ω and point P lies outside the circle. The given points are such that (i) lines PB and PD are tangent to ω, (ii) P, A, C are collinear, and (iii) $\overline{DE} \parallel \overline{AC}$.

Prove that $\overline{BE}$ bisects $\overline{AC}$. **Hints**: 401 575 **Sol:** p.242

Lemma 1.44 (Three Tangents). *Let ABC be an acute triangle. Let $\overline{BE}$ and $\overline{CF}$ be altitudes of $\triangle ABC$, and denote by M the midpoint of $\overline{BC}$. Prove that $\overline{ME}$, $\overline{MF}$, and the line through A parallel to $\overline{BC}$ are all tangents to (AEF).* **Hints**: 24 335

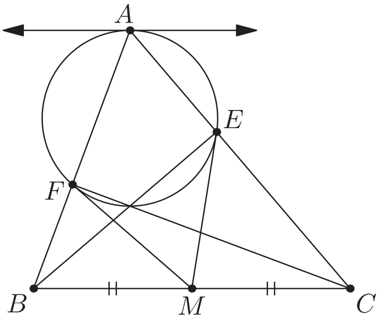

Figure 1.8B. Lemma 1.44, involving tangents to (AEF).

Lemma 1.45 (Right Angles on Incircle Chord). *The incircle of $\triangle ABC$ is tangent to $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ at D, E, F, respectively. Let M and N be the midpoints of $\overline{BC}$ and $\overline{AC}$, respectively. Ray BI meets line EF at K. Show that $\overline{BK} \perp \overline{CK}$. Then show K lies on line MN.* **Hints**: 460 84

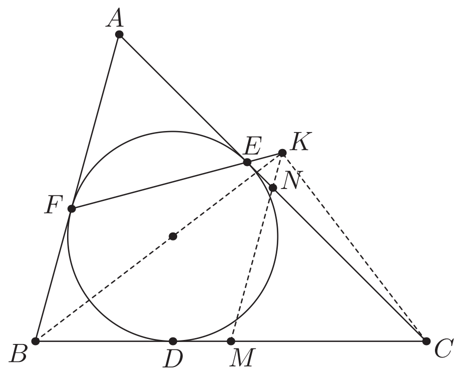

Figure 1.8C. Diagram for Lemma 1.45.

Problem 1.46 (Canada 1997/4). The point O is situated inside the parallelogram $ABCD$ such that $\angle AOB + \angle COD = 180^\circ$. Prove that $\angle OBC = \angle ODC$. **Hints**: 386 110 214 **Sol:** p.242

Problem 1.47 (IMO 2006/1). Let ABC be triangle with incenter I. A point P in the interior of the triangle satisfies

$$\angle PBA + \angle PCA = \angle PBC + \angle PCB.$$

Show that $AP \geq AI$ and that equality holds if and only if $P = I$. **Hints**: 212 453 670

Lemma 1.48 (Simson Line). *Let ABC be a triangle and P be any point on (ABC). Let X, Y, Z be the feet of the perpendiculars from P onto lines BC, CA, and AB. Prove that points X, Y, Z are collinear.* **Hints**: 278 502 **Sol:** p.243

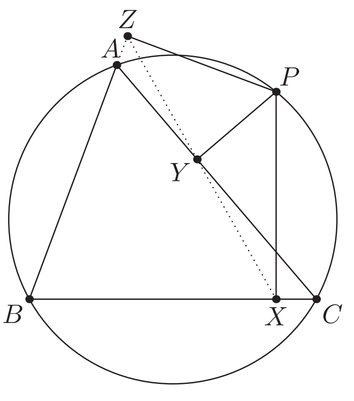

Figure 1.8D. Lemma 1.48; the **Simson line**.

Problem 1.49 (USAMO 2010/1). Let $AXYZB$ be a convex pentagon inscribed in a semicircle of diameter AB. Denote by P, Q, R, S the feet of the perpendiculars from Y onto lines AX, BX, AZ, BZ, respectively. Prove that the acute angle formed by lines PQ and RS is half the size of $\angle XOZ$, where O is the midpoint of segment AB. **Hint**: 661

Problem 1.50 (IMO 2013/4). Let ABC be an acute triangle with orthocenter H, and let W be a point on the side $\overline{BC}$, between B and C. The points M and N are the feet of the altitudes drawn from B and C, respectively. ω_1 is the circumcircle of triangle BWN and X is a point such that $\overline{WX}$ is a diameter of ω_1. Similarly, ω_2 is the circumcircle of triangle CWM and Y is a point such that $\overline{WY}$ is a diameter of ω_2. Show that the points X, Y, and H are collinear. **Hints**: 106 157 15 **Sol:** p.243

Problem 1.51 (IMO 1985/1). A circle has center on the side $\overline{AB}$ of the cyclic quadrilateral $ABCD$. The other three sides are tangent to the circle. Prove that $AD + BC = AB$. **Hints**: 36 201

CHAPTER 2

Circles

Construct a circle of radius zero. . .

Although it is often an intermediate step, angle chasing is usually not enough to solve a problem completely. In this chapter, we develop some other fundamental tools involving circles.

2.1 Orientations of Similar Triangles

You probably already know the similarity criterion for triangles. Similar triangles are useful because they let us convert angle information into lengths. This leads to the power of a point theorem, arguably the most common sets of similar triangles.

In preparation for the upcoming section, we develop the notion of similar triangles that are similarly oriented and oppositely oriented.

Here is how it works. Consider triangles ABC and XYZ. We say they are **directly similar**, or similar and **similarly oriented**, if

$$\measuredangle ABC = \measuredangle XYZ, \ \measuredangle BCA = \measuredangle YZX, \ \text{and } \measuredangle CAB = \measuredangle ZXY.$$

We say they are **oppositely similar**, or similar and **oppositely oriented**, if

$$\measuredangle ABC = -\measuredangle XYZ, \ \measuredangle BCA = -\measuredangle YZX, \ \text{and } \measuredangle CAB = -\measuredangle ZXY.$$

If they are either directly similar or oppositely similar, then they are **similar**. We write $\triangle ABC \sim \triangle XYZ$ in this case. See Figure 2.1A for an illustration.

Two of the angle equalities imply the third, so this is essentially directed AA. Remember to pay attention to the order of the points.

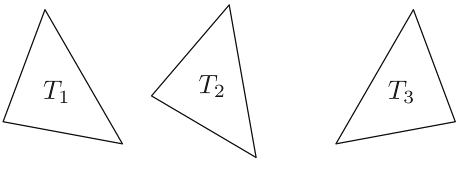

Figure 2.1A. T_1 is directly similar to T_2 and oppositely to T_3.

The upshot of this is that we may continue to use directed angles when proving triangles are similar; we just need to be a little more careful. In any case, as you probably already know, similar triangles also produce ratios of lengths.

Proposition 2.1 (Similar Triangles). *The following are equivalent for triangles ABC and XYZ.*

(i) $\triangle ABC \sim \triangle XYZ$.
(ii) (AA) $\angle A = \angle X$ *and* $\angle B = \angle Y$.
(iii) (SAS) $\angle B = \angle Y$, *and* $AB : XY = BC : YZ$.
(iv) (SSS) $AB : XY = BC : YZ = CA : ZX$.

Thus, lengths (particularly their ratios) can induce similar triangles and vice versa. It is important to notice that SAS similarity does not have a directed form; see Problem 2.2. In the context of angle chasing, we are interested in showing that two triangles are similar using directed AA, and then using the resulting length information to finish the problem. The power of a point theorem in the next section is perhaps the greatest demonstration. However, we remind the reader that angle chasing is only a small part of olympiad geometry, and not to overuse it.

Problem for this Section

Problem 2.2. Find an example of two triangles ABC and XYZ such that $AB : XY = BC : YZ$, $\measuredangle BCA = \measuredangle YZX$, but $\triangle ABC$ and $\triangle XYZ$ are not similar.

2.2 Power of a Point

Cyclic quadrilaterals have many equal angles, so it should come as no surprise that we should be able to find some similar triangles. Let us see what length relations we can deduce.

Consider four points A, B, X, Y lying on a circle. Let line AB and line XY intersect at P. See Figure 2.2A.

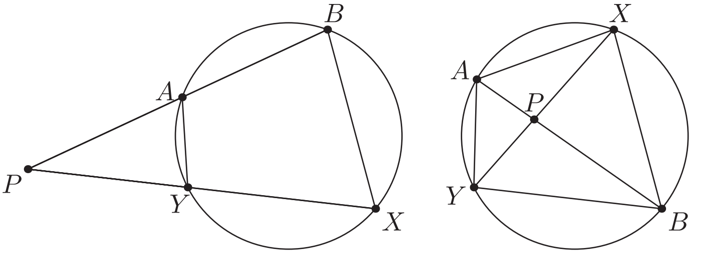

Figure 2.2A. Configurations in power of a point.

A simple directed angle chase gives that

$$\measuredangle PAY = \measuredangle BAY = \measuredangle BXY = \measuredangle BXP = -\measuredangle PXB$$

and

$$\measuredangle AYP = \measuredangle AYX = \measuredangle ABX = \measuredangle PBX = -\measuredangle XBP.$$

As a result, we deduce that $\triangle PAY$ is oppositely similar to $\triangle PXB$.

Therefore, we derive

$$\frac{PA}{PY} = \frac{PX}{PB}$$

or

$$PA \cdot PB = PX \cdot PY.$$

This is the heart of the theorem. Another way to think of this is that the quantity $PA \cdot PB$ does not depend on the choice of line AB, but instead only on the point P. In particular, if we choose line AB to pass through the center of the circle, we obtain that $PA \cdot PB = |PO - r||PO + r|$ where O and r are the center and radius of ω, respectively. In light of this, we define the **power of P** with respect to the circle ω by

$$\operatorname{Pow}_\omega(P) = OP^2 - r^2.$$

This quantity may be negative. Actually, the sign allows us to detect whether P lies inside the circle or not. With this definition we obtain the following properties.

Theorem 2.3 (Power of a Point). *Consider a circle ω and an arbitrary point P.*

(a) The quantity $\operatorname{Pow}_\omega(P)$ is positive, zero, or negative according to whether P is outside, on, or inside ω, respectively.

(b) If ℓ is a line through P intersecting ω at two distinct points X and Y, then

$$PX \cdot PY = |\operatorname{Pow}_\omega(P)|.$$

(c) If P is outside ω and $\overline{PA}$ is a tangent to ω at a point A on ω, then

$$PA^2 = \operatorname{Pow}_\omega(P).$$

Perhaps even more important is the converse of the power of a point, which allows us to find cyclic quadrilaterals based on length. Here it is.

Theorem 2.4 (Converse of the Power of a Point). *Let A, B, X, Y be four distinct points in the plane and let lines AB and XY intersect at P. Suppose that either P lies in both of the segments $\overline{AB}$ and $\overline{XY}$, or in neither segment. If $PA \cdot PB = PX \cdot PY$, then A, B, X, Y are concyclic.*

Proof. The proof is by phantom points (see Example 1.32, say). Let line XP meet (ABX) at Y'. Then A, B, X, Y' are concyclic. Therefore, by power of a point, $PA \cdot PB = PX \cdot PY'$. Yet we are given $PA \cdot PB = PX \cdot PY$. This implies $PY = PY'$.

We are not quite done! We would like that $Y = Y'$, but $PY = PY'$ is not quite enough. See Figure 2.2B. It is possible that Y and Y' are reflections across point P.

Fortunately, the final condition now comes in. Assume for the sake of contradiction that $Y \neq Y'$; then Y and Y' are reflections across P. The fact that A, B, X, Y' are concyclic implies that P lies in both or neither of $\overline{AB}$ and $\overline{XY'}$. Either way, this changes if we consider $\overline{AB}$ and $\overline{XY}$. This violates the second hypothesis of the theorem, contradiction. $\square$

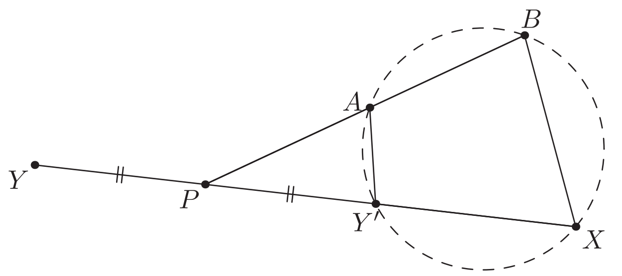

Figure 2.2B. It's a trap! $PA \cdot PB = PX \cdot PY$ almost implies concyclic, but not quite.

As you might guess, the above theorem often provides a bridge between angle chasing and lengths. In fact, it can appear in even more unexpected ways. See the next section.

Problems for this Section

Problem 2.5. Prove Theorem 2.3.

Problem 2.6. Let ABC be a right triangle with $\angle ACB = 90^\circ$. Give a proof of the Pythagorean theorem using Figure 2.2C. (Make sure to avoid a circular proof.)

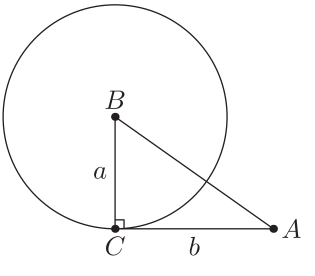

Figure 2.2C. A proof of the Pythagorean theorem.

2.3 The Radical Axis and Radical Center

We start this section with a teaser.

Example 2.7. Three circles intersect as in Figure 2.3A. Prove that the common chords are concurrent.

This seems totally beyond the reach of angle chasing, and indeed it is. The key to unlocking this is the radical axis.

Given two circles ω_1 and ω_2 with distinct centers, the **radical axis** of the circles is the set of points P such that

$$\text{Pow}_{\omega_1}(P) = \text{Pow}_{\omega_2}(P).$$

At first, this seems completely arbitrary. What could possibly be interesting about having equal power to two circles? Surprisingly, the situation is almost the opposite.

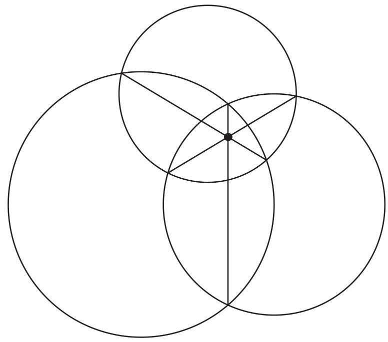

Figure 2.3A. The common chords are concurrent.

Theorem 2.8 (Radical Axis). *Let ω_1 and ω_2 be circles with distinct centers O_1 and O_2. The radical axis of ω_1 and ω_2 is a straight line perpendicular to $\overline{O_1O_2}$.*

In particular, if ω_1 and ω_2 intersect at two points A and B, then the radical axis is line AB.

An illustration is in Figure 2.3B.

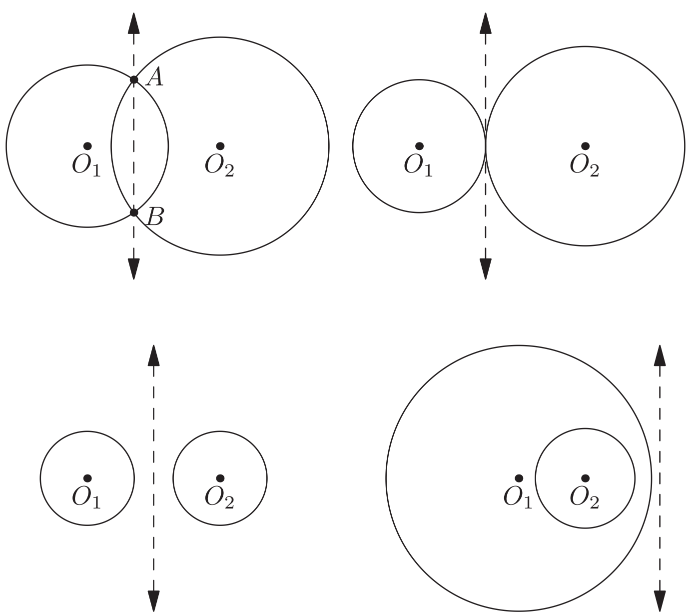

Figure 2.3B. Radical axes on display.

Proof. This is one of the nicer applications of Cartesian coordinates—we are motivated to do so by the squares of lengths appearing, and the perpendicularity of the lines. Suppose that $O_1 = (a, 0)$ and $O_2 = (b, 0)$ in the coordinate plane and the circles have radii r_1 and r_2 respectively. Then for any point $P = (x, y)$ we have

$$\operatorname{Pow}_{\omega_1}(P) = O_1P^2 - r_1^2 = (x-a)^2 + y^2 - r_1^2.$$

Similarly,

$$\mathrm{Pow}_{\omega_2}(P) = O_2P^2 - r_2^2 = (x-b)^2 + y^2 - r_2^2.$$

Equating the two, we find the radical axis of ω_1 and ω_2 is the set of points $P = (x, y)$ satisfying

$$\begin{aligned} 0 &= \mathrm{Pow}_{\omega_1}(P) - \mathrm{Pow}_{\omega_2}(P) \\ &= \left[(x-a)^2 + y^2 - r_1^2\right] - \left[(x-b)^2 + y^2 - r_2^2\right] \\ &= (-2a+2b)x + \left(a^2 - b^2 + r_2^2 - r_1^2\right) \end{aligned}$$

which is a straight line perpendicular to the x-axis (as $-2a + 2b \neq 0$). This implies the result.

The second part is an immediately corollary. The points A and B have equal power (namely zero) to both circles; therefore, both A and B lie on the radical axis. Consequently, the radical axis must be the line AB itself. □

As a side remark, you might have realized in the proof that the standard equation of a circle $(x-m)^2 + (y-n)^2 - r^2 = 0$ is actually just the expansion of $\mathrm{Pow}_\omega((x, y)) = 0$. That is, the expression $(x-m)^2 + (y-n)^2 - r^2$ actually yields the power of the point (x, y) in Cartesian coordinates to the circle centered at (m, n) with radius r.

The power of Theorem 2.8 (no pun intended) is the fact that it is essentially an "if and only if" statement. That is, a point has equal power to both circles if and only if it lies on the radical axis, which we know much about.

Let us now return to the problem we saw at the beginning of this section. Some of you may already be able to guess the ending.

Proof of Example 2.7. The common chords are radical axes. Let ℓ_{12} be the radical axis of ω_1 and ω_2, and let ℓ_{23} be the radical axis of ω_2 and ω_3.

Let P be the intersection of these two lines. Then

$$P \in \ell_{12} \Rightarrow \mathrm{Pow}_{\omega_1}(P) = \mathrm{Pow}_{\omega_2}(P)$$

and

$$P \in \ell_{23} \Rightarrow \mathrm{Pow}_{\omega_2}(P) = \mathrm{Pow}_{\omega_3}(P)$$

which implies $\mathrm{Pow}_{\omega_1}(P) = \mathrm{Pow}_{\omega_3}(P)$. Hence $P \in \ell_{31}$ and accordingly we discover that all three lines pass through P. □

In general, consider three circles with distinct centers O_1, O_2, O_3. In light of the discussion above, there are two possibilities.

1. Usually, the pairwise radical axes concur at a single point K. In that case, we call K the **radical center** of the three circles.
2. Occasionally, the three radical axes will be pairwise parallel (or even the same line). Because the radical axis of two circles is perpendicular to the line joining its centers, this (annoying) case can only occur if O_1, O_2, O_3 are collinear.

It is easy to see that these are the only possibilities; whenever two radical axes intersect, then the third one must pass through their intersection point.

We should also recognize that the converse to Example 2.7 is also true. Here is the full configuration.

Theorem 2.9 (Radical Center of Intersecting Circles). *Let ω_1 and ω_2 be two circles with centers O_1 and O_2. Select points A and B on ω_1 and points C and D on ω_2. Then the following are equivalent:*

(a) A, B, C, D lie on a circle with center O_3 not on line O_1O_2.
(b) Lines AB and CD intersect on the radical axis of ω_1 and ω_2.

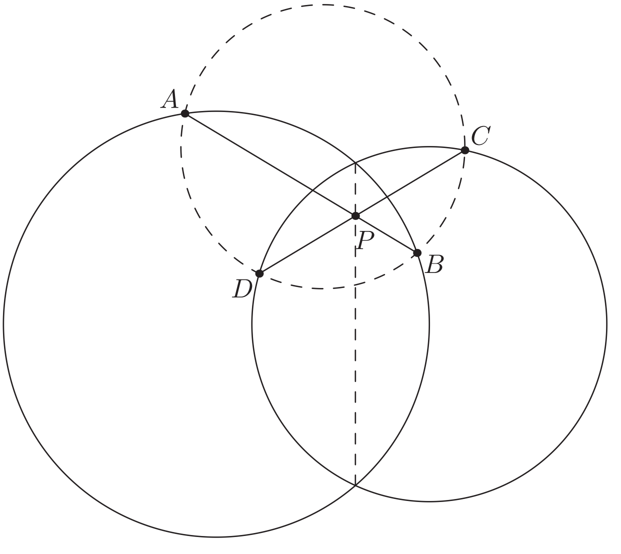

Figure 2.3C. The converse is also true. See Theorem 2.9.

Proof. We have already shown one direction. Now suppose lines AB and CD intersect at P, and that P lies on the radical axis. Then

$$\pm PA \cdot PB = \operatorname{Pow}_{\omega_1}(P) = \operatorname{Pow}_{\omega_2}(P) = \pm PC \cdot PD.$$

We need one final remark: we see that $\operatorname{Pow}_{\omega_1}(P) > 0$ if and only if P lies strictly between A and B. Similarly, $\operatorname{Pow}_{\omega_2}(P) > 0$ if and only if P lies strictly between C and D. Because $\operatorname{Pow}_{\omega_1}(P) = \operatorname{Pow}_{\omega_2}(P)$, we have the good case of Theorem 2.4. Hence, because $PA \cdot PB = PC \cdot PD$, we conclude that A, B, C, D are concyclic. Because lines AB and CD are not parallel, it must also be the case that the points O_1, O_2, O_3 are not collinear. $\square$

We have been very careful in our examples above to check that the power of a point holds in the right direction, and to treat the two cases "concurrent" or "all parallel". In practice, this is more rarely an issue, because the specific configuration in an olympiad problem often excludes such pathological configurations. Perhaps one notable exception is USAMO 2009/1 (Example 2.21).

To conclude this section, here is one interesting application of the radical axis that is too surprising to be excluded.

Proposition 2.10. *In a triangle ABC, the circumcenter exists. That is, there is a point O such that $OA = OB = OC$.*

Proof. Construct a circle of radius zero (!) centered at A, and denote it by ω_A. Define ω_B and ω_C similarly. Because the centers are not collinear, we can find their radical center O.

Now we know the powers from O to each of ω_A, ω_B, ω_C are equal. Rephrased, the (squared) length of the "tangents" to each circle are equal: that is, $OA^2 = OB^2 = OC^2$. (To see that OA^2 really is the power, just use $\text{Pow}_{\omega_A}(O) = OA^2 - 0^2 = OA^2$.) From here we derive that $OA = OB = OC$, as required. $\square$

Of course, the radical axes are actually just the perpendicular bisectors of the sides. But this presentation was simply too surprising to forgo. This may be the first time you have seen a circle of radius zero; it will not be the last.

Problems for this Section

Lemma 2.11. *Let ABC be a triangle and consider a point P in its interior. Suppose that $\overline{BC}$ is tangent to the circumcircles of triangles ABP and ACP. Prove that ray AP bisects $\overline{BC}$.*

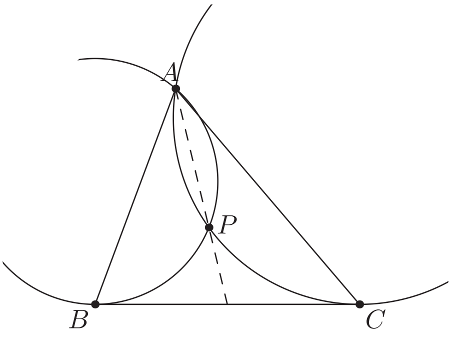

Figure 2.3D. Diagram for Lemma 2.11.

Problem 2.12. Show that the orthocenter of a triangle exists using radical axes. That is, if $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ are altitudes of a triangle ABC, show that the altitudes are concurrent.
Hint: 367

2.4 Coaxial Circles

If a set of circles have the same radical axes, then we say they are **coaxial**. A collection of such circles is called a **pencil** of coaxial circles. In particular, if circles are coaxal, their centers are collinear. (The converse is not true.)

Coaxial circles can arise naturally in the following way.

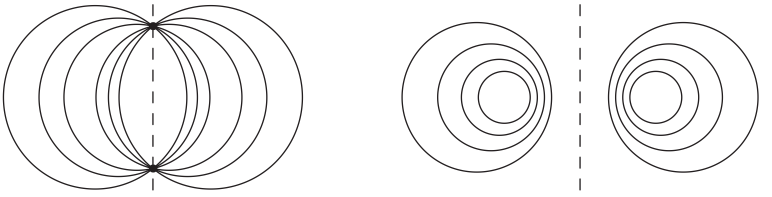

Figure 2.4A. Two pencils of coaxial circles.

Lemma 2.13 (Finding Coaxial Circles). *Three distinct circles Ω_1, Ω_2, Ω_3 pass through a point X. Then their centers are collinear if and only if they share a second common point.*

Proof. Both conditions are equivalent to being coaxial. □

2.5 Revisiting Tangents: The Incenter

We consider again an angle bisector. See Figure 2.5A.

For any point P on the angle bisector, the distances from P to the sides are equal. Consequently, we can draw a circle centered at P tangent to the two sides. Conversely, the two tangents to any circle always have equal length, and the center of that circle lies on the corresponding angle bisector.

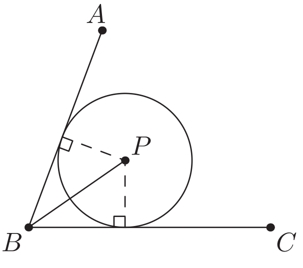

Figure 2.5A. Two tangents to a circle.

From these remarks we can better understand the incenter.

Proposition 2.14. *In any triangle ABC, the angle bisectors concur at a point I, which is the center of a circle inscribed in the triangle.*

Proof. Essentially we are going to complete Figure 2.5A to obtain Figure 2.5B. Let the angle bisectors of $\angle B$ and $\angle C$ intersect at a point I. We claim that I is the desired incenter.

Let D, E, F be the projections of I onto $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, respectively. Because I is on the angle bisector of $\angle B$, we know that $IF = ID$. Because I is on the angle bisector of $\angle C$, we know that $ID = IE$. (If this reminds you of the proof of the radical center, it should!) Therefore, $IE = IF$, and we deduce that I is also on the angle bisector of $\angle A$. Finally, the circle centered at I with radius $ID = IE = IF$ is evidently tangent to all sides. □

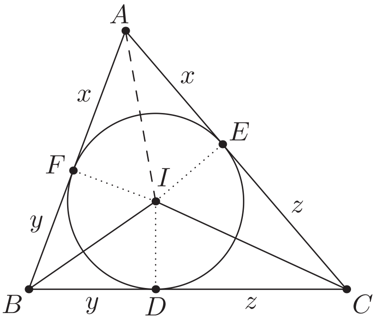

Figure 2.5B. Describing the incircle of a triangle.

The triangle DEF is called the **contact triangle** of $\triangle ABC$.

We can say even more. In Figure 2.5B we have marked the equal lengths induced by the tangents as x, y, and z. Considering each of the sides, this gives us a system of equations of three variables

$$y + z = a$$
$$z + x = b$$
$$x + y = c.$$

Now we can solve for x, y, and z in terms of a, b, c. This is left as an exercise, but we state the result here. (Here $s = \frac{1}{2}(a + b + c)$.)

Lemma 2.15 (Tangents to the Incircle). *If DEF is the contact triangle of $\triangle ABC$, then $AE = AF = s - a$. Similarly, $BF = BD = s - b$ and $CD = CE = s - c$.*

Problem for this Section

Problem 2.16. Prove Lemma 2.15.

2.6 The Excircles

In Lemma 1.18 we briefly alluded the excenter of a triangle. Let us consider it more completely here. The ***A*-excircle** of a triangle ABC is the circle that is tangent to $\overline{BC}$, the extension of $\overline{AB}$ past B, and the extension of $\overline{AC}$ past C. See Figure 2.6A. The ***A*-excenter**, usually denoted I_A, is the center of the A-excircle. The B-excircle and C-excircles are defined similarly and their centers are unsurprisingly called the B-excenter and the C-excenter.

We have to actually check that the A-excircle exists, as it is not entirely obvious from the definition. The proof is exactly analogous to that for the incenter, except with the angle bisector from B replaced with an **external angle bisector**, and similarly for C. As a simple corollary, the incenter of ABC lies on $\overline{AI_A}$.

Now let us see if we can find similar length relations as in the incircle. Let X be the tangency point of the A-excircle on $\overline{BC}$ and B_1 and C_1 the tangency points to rays AB and

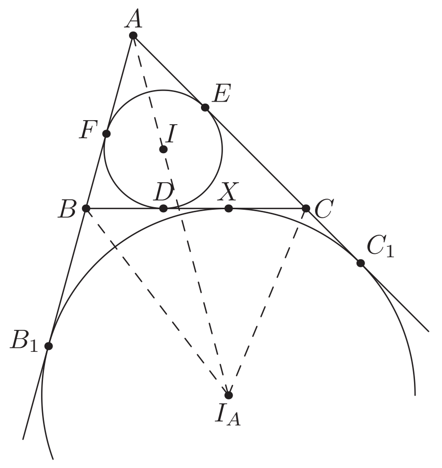

Figure 2.6A. The incircle and A-excircle.

AC. We know that $AB_1 = AC_1$ and that

$$\begin{aligned} AB_1 + AC_1 &= (AB + BB_1) + (AC + CC_1) \\ &= (AB + BX) + (AC + CX) \\ &= AB + AC + BC \\ &= 2s. \end{aligned}$$

We have now obtained the following.

Lemma 2.17 (Tangents to the Excircle). *If AB_1 and AC_1 are the tangents to the A-excircle, then $AB_1 = AC_1 = s$.*

Let us make one last remark: in Figure 2.6A, the triangles AIF and AI_AB_1 are directly similar. (Why?) This lets us relate the ***A*-exradius**, or the radius of the excircle, to the other lengths in the triangle. This exradius is usually denoted r_a. See Lemma 2.19.

Problems for this Section

Problem 2.18. Let the external angle bisectors of B and C in a triangle ABC intersect at I_A. Show that I_A is the center of a circle tangent to $\overline{BC}$, the extension of $\overline{AB}$ through B, and the extension of $\overline{AC}$ through C. Furthermore, show that I_A lies on ray AI.

Lemma 2.19 (Length of Exradius). *Prove that the A-exradius has length*

$$r_a = \frac{s}{s-a} r.$$

Hint: 302

Lemma 2.20. *Let ABC be a triangle. Suppose its incircle and A-excircle are tangent to $\overline{BC}$ at X and D, respectively. Show that $BX = CD$ and $BD = CX$.*

2.7 Example Problems

We finish this chapter with several problems, which we feel are either instructive, classical, or too surprising to not be shared.

Example 2.21 (USAMO 2009/1). Given circles ω_1 and ω_2 intersecting at points X and Y, let ℓ_1 be a line through the center of ω_1 intersecting ω_2 at points P and Q and let ℓ_2 be a line through the center of ω_2 intersecting ω_1 at points R and S. Prove that if P, Q, R, and S lie on a circle then the center of this circle lies on line XY.

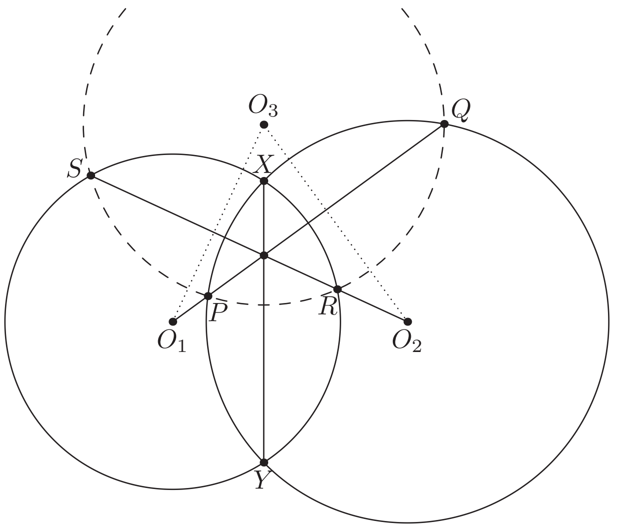

Figure 2.7A. The first problem of the 2009 USAMO.

This was actually a very nasty USAMO problem, in the sense that it was easy to lose partial credit. We will see why.

Let O_3 and ω_3 be the circumcenter and circumcircle, respectively, of the cyclic quadrilateral $PQRS$. After drawing the diagram, we are immediately reminded of our radical axes. In fact, we already know that that lines PQ, RS, and XY concur at a point X, by Theorem 2.9. Call this point H.

Now, what else do we know? Well, glancing at the diagram* it appears that $\overline{O_1O_3} \perp \overline{RS}$. And of course this we know is true, because $\overline{RS}$ is the radical axis of ω_1 an ω_3. Similarly, we notice that $\overline{PQ}$ is perpendicular to O_1O_3.

Focus on $\triangle O_1O_2O_3$. We see that H is its orthocenter. Therefore the altitude from O_3 to $\overline{O_1O_2}$ must pass through H. But line XY is precisely that altitude: it passes through H and is perpendicular to $\overline{O_1O_2}$. Hence, O_3 lies on line XY, and we are done.

Or are we?

Look at Theorem 2.9 again. In order to apply it, we need to know that O_1, O_2, O_3 are not collinear. Unfortunately, this is not always true—see Figure 2.7B.

Fortunately, noticing this case is much harder than actually doing it. We use phantom points. Let O be the midpoint of $\overline{XY}$. (We pick this point because we know this is where O_3

* And you are drawing large scaled diagrams, right?

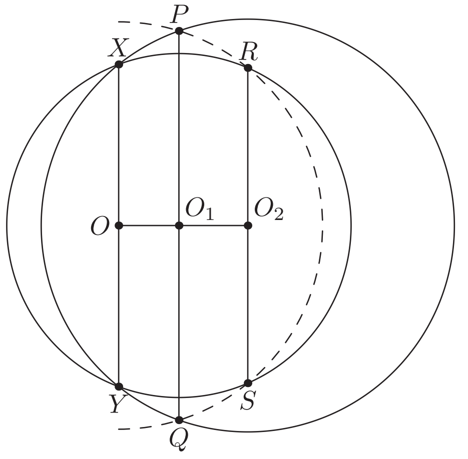

Figure 2.7B. An unnoticed special case.

must be for the problem to hold.) Now we just need to show that $OP = OQ = OR = OS$, from which it will follow that $O = O_3$.

This looks much easier. It should seem like we should be able to compute everything using just repeated applications of the Pythagorean theorem (and the definition of a circle). Trying this,

$$\begin{aligned} OP^2 &= OO_1^2 + O_1P^2 \\ &= OO_1^2 + (O_2P^2 - O_1O_2^2) \\ &= OO_1^2 + r_2^2 - O_1O_2^2. \end{aligned}$$

Now the point P is gone from the expression, but the r_2 needs to go if we hope to get a symmetric expression. We can get rid of it by using $O_2X = r_2 = \sqrt{XO^2 + OO_2^2}$.

$$\begin{aligned} OP^2 &= OO_1^2 + (O_2X^2 + OX^2) - O_1O_2^2 \\ &= OX^2 + OO_1^2 + OO_2^2 - O_1O_2^2 \\ &= \left(\frac{1}{2}XY\right)^2 + OO_1^2 + OO_2^2 - O_1O_2^2. \end{aligned}$$

This is symmetric; the exact same calculations with Q, R, and S yield the same results. We conclude $OP^2 = OQ^2 = OR^2 = OS^2 = \left(\frac{1}{2}XY\right)^2 + OO_1^2 + OO_2^2 - O_1O_2^2$ as desired.

Having presented the perhaps more natural solution above, here is a solution with a more analytic flavor. It carefully avoids the configuration issues in the first solution.

Solution to Example 2.21. Let r_1, r_2, r_3 denote the circumradii of ω_1, ω_2, and ω_3, respectively.

We wish to show that O_3 lies on the radical axis of ω_1 and ω_2. Let us encode the conditions using power of a point. Because O_1 is on the radical axis of ω_2 and ω_3,

$$\begin{aligned}
\mathrm{Pow}_{\omega_2}(O_1) &= \mathrm{Pow}_{\omega_3}(O_1) \\
\Rightarrow O_1O_2^2 - r_2^2 &= O_1O_3^2 - r_3^2.
\end{aligned}$$

Similarly, because O_2 is on the radical axis of ω_1 and ω_3, we have

$$\begin{aligned}
\mathrm{Pow}_{\omega_1}(O_2) &= \mathrm{Pow}_{\omega_3}(O_2) \\
\Rightarrow O_1O_2^2 - r_1^2 &= O_2O_3^2 - r_3^2.
\end{aligned}$$

Subtracting the two gives

$$\begin{aligned}
(O_1O_2^2 - r_2^2) - (O_1O_2^2 - r_1^2) &= (O_1O_3^2 - r_3^2) - (O_2O_3^2 - r_3^2) \\
\Rightarrow r_1^2 - r_2^2 &= O_1O_3^2 - O_2O_3^2 \\
\Rightarrow O_2O_3^2 - r_2^2 &= O_1O_3^2 - r_1^2 \\
\Rightarrow \mathrm{Pow}_{\omega_2}(O_3) &= \mathrm{Pow}_{\omega_1}(O_3)
\end{aligned}$$

as desired. $\square$

The main idea of this solution is to encode everything in terms of lengths using the radical axis. Effectively, we write down the givens as equations. We also write the desired conclusion as an equation, namely $\mathrm{Pow}_{\omega_2}(O_3) = \mathrm{Pow}_{\omega_1}(O_3)$, then forget about geometry and do algebra. It is an unfortunate irony of olympiad geometry that analytic solutions are often immune to configuration issues that would otherwise plague traditional solutions.

The next example is a classical result of Euler.

Lemma 2.22 (Euler's Theorem). *Let ABC be a triangle. Let R and r denote its circumradius and inradius, respectively. Let O and I denote its circumcenter and incenter. Then $OI^2 = R(R-2r)$. In particular, $R \ge 2r$.*

The first thing we notice is that the relation is equivalent to proving $R^2 - OI^2 = 2Rr$. This is power of a point, clear as day. So, we let ray AI hit the circumcircle again at L. Evidently we just need to show

$$AI \cdot IL = 2Rr.$$

This looks much nicer to work with—noticing the power expressions gave us a way to clean up the problem statement, and gives us some structure to work on.

We work backwards for a little bit. The final condition appears like similar triangles. So perhaps we may rewrite it as

$$\frac{AI}{r} = \frac{2R}{IL}.$$

There are not too many ways the left-hand side can show up like that. We drop the altitude from I to $\overline{AB}$ as F. Then $\triangle AIF$ has the ratios that we want. (You can also drop the foot to

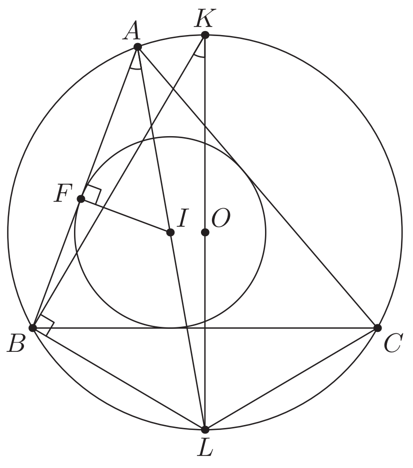

Figure 2.7C. Proving Euler's theorem.

$\overline{AC}$, but this is the same thing.) All that remains is to construct a similar triangle with the lengths $2R$ and IL. Unfortunately, $\overline{IL}$ does not play well in this diagram.

But we hope that by now you recognize $\overline{IL}$ from Lemma 1.18! Write $BL = IL$. Then let K be the point such that $\overline{KL}$ is a diameter of the circle. Then $\triangle KBL$ has the dimensions we want. Could the triangles in question be similar? Yes: $\angle KBL$ and $\angle AFI$ are both right angles, and $\angle BAL = \angle BKL$ by cyclic quadrilaterals. Hence this produces $AI \cdot IL = 2Rr$ and we are done.

As usual, this is not how a solution should be written up in a contest. Solutions should be only written forwards, and without explaining where the steps come from.

Solution to Lemma 2.22. Let ray AI meet the circumcircle again at L and let K be the point diametrically opposite L. Let F be the foot from I to $\overline{AB}$. Notice that $\angle FAI = \angle BAL = \angle BKL$ and $\angle AFI = \angle KBL = 90^\circ$, so

$$\frac{AI}{r} = \frac{AI}{IF} = \frac{KL}{LB} = \frac{2R}{LI}$$

and hence $AI \cdot IL = 2Rr$. Because I lies inside $\triangle ABC$, we deduce the power of I with respect to (ABC) is $2Rr = R^2 - OI^2$. Consequently, $OI^2 = R(R - 2r)$. $\square$

The construction of the diameter appears again in Chapter 3, when we derive the extended law of sines, Theorem 3.1.

Our last example is from the All-Russian Mathematical Olympiad, whose solution is totally unexpected. Please ponder it before reading the solution.

Example 2.23 (Russian Olympiad 2010). Triangle ABC has perimeter 4. Points X and Y lie on rays AB and AC, respectively, such that $AX = AY = 1$. Segments BC and XY intersect at point M. Prove that the perimeter of either $\triangle ABM$ or $\triangle ACM$ is 2.

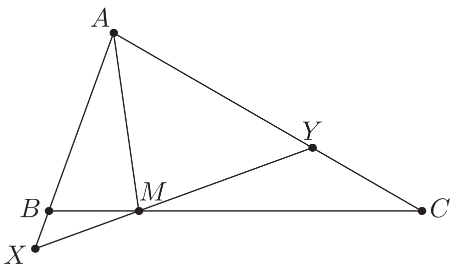

Figure 2.7D. A problem from the All-Russian MO 2010.

What strange conditions have been given. We are told the lengths $AX = AY = 1$ and the perimeter of $\triangle ABC$ is 4, and effectively nothing else. The conclusion, which is an either-or statement, is equally puzzling.

Let us reflect the point A over both X and Y to two points U and V so that $AU = AV = 2$. This seems slightly better, because $AU = AV = 2$ now, and the "two" in the perimeter is now present. But what do we do? Recalling that $s = 2$ in the triangle, we find that U and V are the tangency points of the excircle, call it Γ_a. Set I_A the excenter, tangent to $\overline{BC}$ at T. See Figure 2.7E.

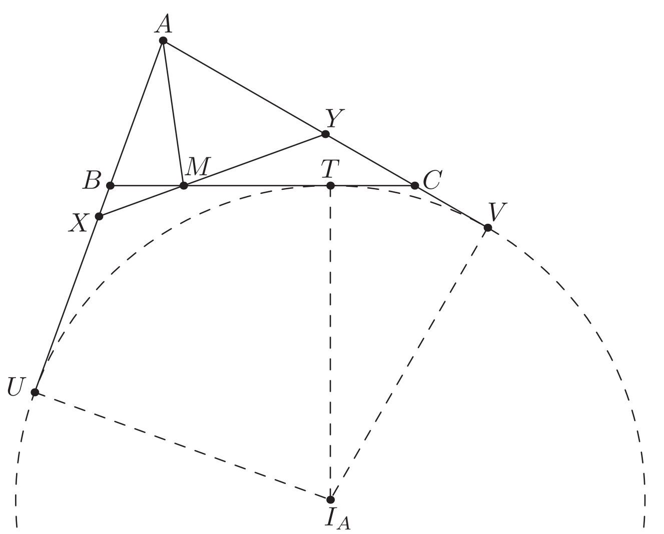

Figure 2.7E. Adding an excircle to handle the conditions.

Looking back, we have now encoded the $AX = AY = 1$ condition as follows: X and Y are the midpoints of the tangents to the A-excircle. We need to show that one of $\triangle ABM$ or $\triangle ACM$ has perimeter equal to the length of the tangent.

Now the question is: how do we use this?

Let us look carefully again at the diagram. It would seem to suggest that in this case, $\triangle ABM$ is the one with perimeter two (and not $\triangle ACM$). What would have to be true in order to obtain the relation $AB + BM + MA = AU$? Trying to bring the lengths closer to the triangle in question, we write $AU = AB + BU = AB + BT$. So we would need $BM + MA = BT$, or $MA = MT$.

So it would appear that the points X, M, Y have the property that their distance to A equals the length of their tangents to the A-excircle. This motivates a last addition to our diagram: construct a circle of radius zero at A, say ω_0. Then X and Y lie on the radical axis of ω_0 and Γ_a; hence so does M! Now we have $MA = MT$, as required.

Now how does the either-or condition come in? Now it is clear: it reflects whether T lies on $\overline{BM}$ or $\overline{CM}$. (It must lie in at least one, because we are told that M lies inside the segment $\overline{BC}$, and the tangency points of the A-excircle to $\overline{BC}$ always lie in this segment as well.) This completes the solution, which we present concisely below.

Solution to Example 2.23. Let I_A be the center of the A-excircle, tangent to $\overline{BC}$ at T, and to the extensions of $\overline{AB}$ and $\overline{AC}$ at U and V. We see that $AU = AV = s = 2$. Then $\overline{XY}$ is the radical axis of the A-excircle and the circle of radius zero at A. Therefore $AM = MT$.

Assume without loss of generality that T lies on $\overline{MC}$, as opposed to $\overline{MB}$. Then $AB + BM + MA = AB + BM + MT = AB + BT = AB + BU = AU = 2$ as desired. □

While we have tried our best to present the solution in a natural way, it is no secret that this is a hard problem by any standard. It is fortunate that such pernicious problems are rare.

2.8 Problems

Lemma 2.24. *Let ABC be a triangle with I_A, I_B, and I_C as excenters. Prove that triangle $I_AI_BI_C$ has orthocenter I and that triangle ABC is its orthic triangle.* **Hints**: 564 103

Theorem 2.25 (The Pitot Theorem). *Let $ABCD$ be a quadrilateral. If a circle can be inscribed*[†] *in it, prove that $AB + CD = BC + DA$.* **Hint**: 467

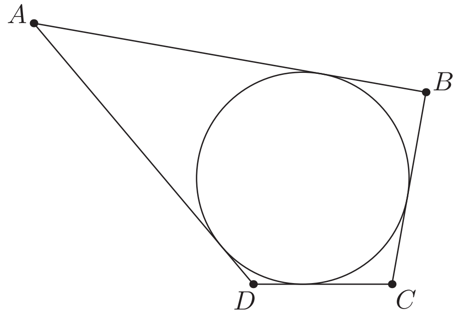

Figure 2.8A. The Pitot theorem: $AB + CD = BC + DA$.

[†] The converse of the Pitot theorem is in fact also true: if $AB + CD = BC + DA$, then a circle can be inscribed inside $ABCD$. Thus, if you ever need to prove $AB + CD = BC + DA$, you may safely replace this with the "inscribed" condition.

Problem 2.26 (USAMO 1990/5). An acute-angled triangle ABC is given in the plane. The circle with diameter $\overline{AB}$ intersects altitude $\overline{CC'}$ and its extension at points M and N, and the circle with diameter $\overline{AC}$ intersects altitude $\overline{BB'}$ and its extensions at P and Q. Prove that the points M, N, P, Q lie on a common circle. **Hints**: 260 73 409 **Sol:** p.244

Problem 2.27 (BAMO 2012/4). Given a segment $\overline{AB}$ in the plane, choose on it a point M different from A and B. Two equilateral triangles AMC and BMD in the plane are constructed on the same side of segment $\overline{AB}$. The circumcircles of the two triangles intersect in point M and another point N.

(a) Prove that $\overline{AD}$ and $\overline{BC}$ pass through point N. **Hints**: 57 77
(b) Prove that no matter where one chooses the point M along segment $\overline{AB}$, all lines MN will pass through some fixed point K in the plane. **Hints**: 230 654

Problem 2.28 (JMO 2012/1). Given a triangle ABC, let P and Q be points on segments $\overline{AB}$ and $\overline{AC}$, respectively, such that $AP = AQ$. Let S and R be distinct points on segment $\overline{BC}$ such that S lies between B and R, $\angle BPS = \angle PRS$, and $\angle CQR = \angle QSR$. Prove that P, Q, R, S are concyclic. **Hints**: 435 601 537 122

Problem 2.29 (IMO 2008/1). Let H be the orthocenter of an acute-angled triangle ABC. The circle Γ_A centered at the midpoint of $\overline{BC}$ and passing through H intersects the sideline BC at points A_1 and A_2. Similarly, define the points B_1, B_2, C_1, and C_2. Prove that six points A_1, A_2, B_1, B_2, C_1, and C_2 are concyclic. **Hints**: 82 597 **Sol:** p.244

Problem 2.30 (USAMO 1997/2). Let ABC be a triangle. Take points D, E, F on the perpendicular bisectors of $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ respectively. Show that the lines through A, B, C perpendicular to $\overline{EF}$, $\overline{FD}$, $\overline{DE}$ respectively are concurrent. **Hints**: 596 2 611

Problem 2.31 (IMO 1995/1). Let A, B, C, D be four distinct points on a line, in that order. The circles with diameters $\overline{AC}$ and $\overline{BD}$ intersect at X and Y. The line XY meets $\overline{BC}$ at Z. Let P be a point on the line XY other than Z. The line CP intersects the circle with diameter AC at C and M, and the line BP intersects the circle with diameter BD at B and N. Prove that the lines AM, DN, XY are concurrent. **Hints**: 49 159 134

Problem 2.32 (USAMO 1998/2). Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be concentric circles, with $\mathcal{C}_2$ in the interior of $\mathcal{C}_1$. From a point A on $\mathcal{C}_1$ one draws the tangent $\overline{AB}$ to $\mathcal{C}_2$ ($B \in \mathcal{C}_2$). Let C be the second point of intersection of ray AB and $\mathcal{C}_1$, and let D be the midpoint of $\overline{AB}$. A line passing through A intersects $\mathcal{C}_2$ at E and F in such a way that the perpendicular bisectors of DE and CF intersect at a point M on AB. Find, with proof, the ratio AM/MC. **Hints**: 659 355 482

Problem 2.33 (IMO 2000/1). Two circles G_1 and G_2 intersect at two points M and N. Let AB be the line tangent to these circles at A and B, respectively, so that M lies closer to AB than N. Let CD be the line parallel to AB and passing through the point M, with C on G_1 and D on G_2. Lines AC and BD meet at E; lines AN and CD meet at P; lines BN and CD meet at Q. Show that $EP = EQ$. **Hints**: 17 174

Problem 2.34 (Canada 1990/3). Let $ABCD$ be a cyclic quadrilateral whose diagonals meet at P. Let W, X, Y, Z be the feet of P onto $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$, respectively. Show that $WX + YZ = XY + WZ$. **Hints**: 1 414 440 **Sol:** p.245

Problem 2.35 (IMO 2009/2). Let ABC be a triangle with circumcenter O. The points P and Q are interior points of the sides $\overline{CA}$ and $\overline{AB}$, respectively. Let K, L, and M be the midpoints of the segments BP, CQ, and PQ, respectively, and let Γ be the circle passing through K, L, and M. Suppose that the line PQ is tangent to the circle Γ. Prove that $OP = OQ$. **Hints**: 78 544 346

Problem 2.36. Let $\overline{AD}$, $\overline{BE}$, $\overline{CF}$ be the altitudes of a scalene triangle ABC with circumcenter O. Prove that (AOD), (BOE), and (COF) intersect at point X other than O. **Hints**: 553 79 **Sol:** p.245

Problem 2.37 (Canada 2007/5). Let the incircle of triangle ABC touch sides BC, CA, and AB at D, E, and F, respectively. Let ω, ω_1, ω_2, and ω_3 denote the circumcircles of triangles ABC, AEF, BDF, and CDE respectively. Let ω and ω_1 intersect at A and P, ω and ω_2 intersect at B and Q, ω and ω_3 intersect at C and R.

(a) Prove that ω_1, ω_2, and ω_3 intersect in a common point.
(b) Show that lines PD, QE, and RF are concurrent. **Hints**: 376 548 660

Problem 2.38 (Iran TST 2011/1). In acute triangle ABC, $\angle B$ is greater than $\angle C$. Let M be the midpoint of $\overline{BC}$ and let E and F be the feet of the altitudes from B and C, respectively. Let K and L be the midpoints of $\overline{ME}$ and $\overline{MF}$, respectively, and let T be on line KL such that $\overline{TA} \parallel \overline{BC}$. Prove that $TA = TM$. **Hints**: 297 495 154 **Sol:** p.246

CHAPTER 3

Lengths and Ratios

As one, who versed in geometric lore, would fain
Measure the circle — Dante, *The Divine Comedy*

3.1 The Extended Law of Sines

Aside from angles and similar triangles, one way to relate angles to lengths is through the **law of sines**. A more thorough introduction to the true power of trigonometry occurs in Section 5.3, but we see that it already proves useful here in our study of lengths.

Theorem 3.1 (The Extended Law of Sines). *In a triangle ABC with circumradius R, we have*

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$

This so-called "extended form" contains the final clause of $2R$ at the end. It has the advantage that it makes the symmetry more clear (if $\frac{a}{\sin A} = 2R$ is true, then the other parts follow rather immediately). The extended form also gives us a hint of a direct proof:

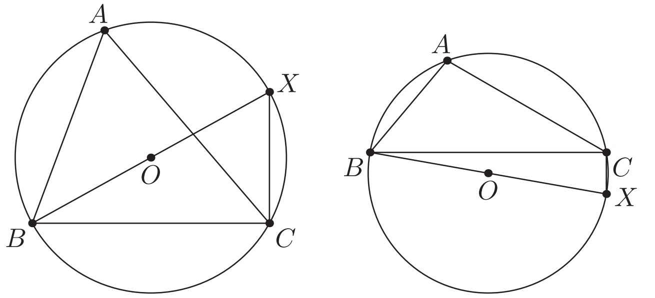

Figure 3.1A. Proving the law of sines.

Proof. As discussed above we only need to prove $\frac{a}{\sin A} = 2R$. Let $\overline{BX}$ be a diameter of the circumcircle, as in Figure 3.1A. Evidently $\measuredangle BXC = \measuredangle BAC$. Now consider triangle

BXC. It is a right triangle with $BC = a$, $BX = 2R$, and either $\angle BXC = A$ or $\angle BXC = 180^\circ - A$ (depending on whether $\angle A$ is acute). Either way,

$$\sin A = \sin \angle BXC = \frac{a}{2R}$$

and the proof ends here. $\square$

The law of sines will be used later to provide a different form of the upcoming Ceva's theorem, namely Theorem 3.4.

Problem for this Section

Theorem 3.2 (Angle Bisector Theorem). *Let ABC be a triangle and D a point on $\overline{BC}$ so that $\overline{AD}$ is the internal angle bisector of $\angle BAC$. Show that*

$$\frac{AB}{AC} = \frac{DB}{DC}.$$

Hint: 417

3.2 Ceva's Theorem

In a triangle, a **cevian** is a line joining a vertex of the triangle to a point on the interior* of the opposite side. A natural question is when three cevians of a triangle are concurrent. This is answered by Ceva's theorem.

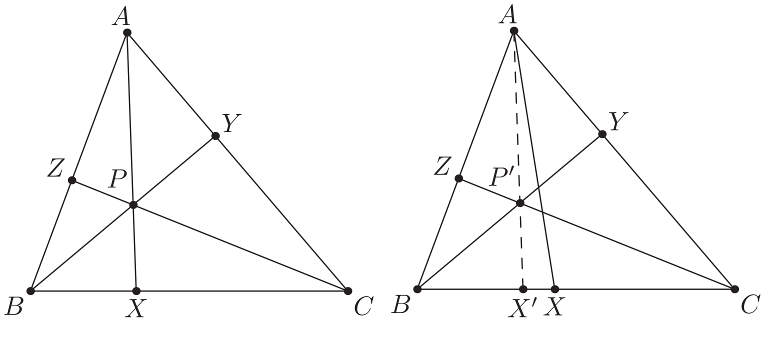

Figure 3.2A. Three cevians are concurrent as in Ceva's theorem.

Theorem 3.3 (Ceva's Theorem). *Let $\overline{AX}, \overline{BY}, \overline{CZ}$ be cevians of a triangle ABC. They concur if and only if*

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1.$$

The proof is by areas: we use the fact that if two triangles share an altitude, the ratio of the areas is the ratio of their bases. This trick is very useful in general.

* Some authors permit cevians to land on points on the *extensions* of the opposite side as well. For this chapter we assume cevians lie in the interior of the triangle unless otherwise specified.

Proof. Let us first assume the cevians concur at P, and try to show the ratios multiply to 1. Since $\triangle BAX$ and $\triangle XAC$ share an altitude, as do $\triangle BPX$ and $\triangle XPC$, we derive

$$\frac{BX}{XC} = \frac{[BAX]}{[XAC]} = \frac{[BPX]}{[XPC]}.$$

Now we are going to use a little algebraic trick: if $\frac{a}{b} = \frac{x}{y}$, then $\frac{a}{b} = \frac{x}{y} = \frac{a+x}{b+y}$. For example, since $\frac{4}{6} = \frac{10}{15}$, both are equal to $\frac{4+10}{6+15} = \frac{14}{21}$. Applying this to the area ratios yields

$$\frac{BX}{XC} = \frac{[BAX] - [BPX]}{[XAC] - [XPC]} = \frac{[BAP]}{[ACP]}.$$

But now the conclusion is imminent, since

$$\frac{CY}{YA} = \frac{[CBP]}{[BAP]} \text{ and } \frac{AZ}{ZB} = \frac{[ACP]}{[CBP]}$$

whence multiplying gives the desired $\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1$.

Now how do we handle the other direction? Dead simple with phantom points. Assume $\overline{AX}, \overline{BY}, \overline{CZ}$ are cevians with

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1.$$

Let $\overline{BY}$ and $\overline{CZ}$ intersect at P', and let ray AP' meet $\overline{BC}$ at X' (right half of Figure 3.2A). By our work already done, we know that

$$\frac{BX'}{X'C} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1.$$

Thus $\frac{BX'}{X'C} = \frac{BX}{XC}$, which is enough to imply $X = X'$. $\square$

The proof above illustrated two useful ideas—the use of area ratios, and the use of phantom points.

As you might guess, Ceva's theorem is extremely useful for showing that three lines are concurrent. It can also be written in a trigonometric form.

Theorem 3.4 (Trigonometric Form of Ceva's Theorem). *Let $\overline{AX}, \overline{BY}, \overline{CZ}$ be cevians of a triangle ABC. They concur if and only if*

$$\frac{\sin\angle BAX \sin\angle CBY \sin\angle ACZ}{\sin\angle XAC \sin\angle YBA \sin\angle ZCB} = 1.$$

The proof is a simple exercise—just use the law of sines.

With this, the existence of the orthocenter, the incenter, and the centroid are all totally straightforward. For the orthocenter[†], we compute

$$\frac{\sin(90^\circ - B)\sin(90^\circ - C)\sin(90^\circ - A)}{\sin(90^\circ - C)\sin(90^\circ - A)\sin(90^\circ - B)} = 1.$$

[†] Actually we need to handle the case where $\triangle ABC$ is obtuse separately, since in that case two of the altitudes fall outside the triangle. We develop the necessary generalization in the next section, when we discuss directed lengths in Menelaus's theorem.

For the incenter, we compute

$$\frac{\sin \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C}{\sin \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C} = 1.$$

We could also have used the angle bisector theorem in the standard form of Ceva's theorem, giving

$$\frac{c}{b}\frac{a}{c}\frac{b}{a} = 1.$$

Finally, for the centroid we have

$$\frac{1}{1}\frac{1}{1}\frac{1}{1} = 1$$

and we no longer have to take the existence of our centers for granted!

Problems for this Section

Problem 3.5. Show the trigonometric form of Ceva holds.

Problem 3.6. Let $\overline{AM}$, $\overline{BE}$, and $\overline{CF}$ be concurrent cevians of a triangle ABC. Show that $\overline{EF} \parallel \overline{BC}$ if and only if $BM = MC$.

3.3 Directed Lengths and Menelaus's Theorem

The analogous form of Ceva's theorem is called Menelaus's theorem, which specifies when three points on the sides of a triangle (or their extensions) are collinear.

Theorem 3.7 (Menelaus's Theorem). *Let X, Y, Z be points on lines BC, CA, AB in a triangle ABC, distinct from its vertices. Then X, Y, Z are collinear if and only if*

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = -1$$

where the lengths are directed.

Here we have introduced ratios of **directed lengths**. Given collinear points A, Z, B, we say that the ratio $\frac{AZ}{ZB}$ is positive if Z lies between A and B, and negative otherwise. (This is much the same idea as the signs we used in defining the power of a point.) We always say explicitly when lengths are taken to be directed.

Notice the similarity to Ceva's theorem. The use of -1 instead of 1 is important—for if X, Y, Z each lie in the interiors of the sides, it is impossible for the three to be collinear!

Essentially the directed lengths are simply encoding two cases of Menelaus's theorem: when either one or three of $\{X, Y, Z\}$ lie outside the corresponding side. It is easy to check that the sign of the directed ratio is negative precisely in these cases.

There are many proofs of Menelaus's theorem that we leave to other sources. The proof we give shows one direction; if the ratios multiply to -1, then the points are collinear. (The other direction then follows using phantom points.) It is inspired by a proof to Monge's theorem (Theorem 3.22), and it is so surprising that we could not resist including it.

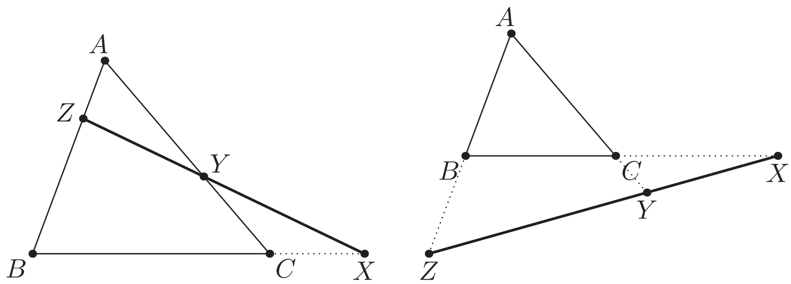

Figure 3.3A. The two cases of Menelaus's theorem.

Proof. First, suppose that the points X, Y, Z lie on the sides of the triangle in such a way that

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = -1.$$

Then it is possible to find nonzero real numbers p, q, r for which

$$\frac{q}{r} = -\frac{BX}{XC}, \quad \frac{r}{p} = -\frac{CY}{YA}, \quad \frac{p}{q} = -\frac{AZ}{ZB}.$$

Now we go into three dimensions! Let $\mathcal{P}$ be the plane of triangle ABC (this page) and construct point A_1 such that $\overline{A_1A} \perp \mathcal{P}$ and $AA_1 = p$; we take A_1 to be above the page if $p > 0$ and below the page otherwise. Now define B_1 and C_1 analogously, so that $BB_1 = q$ and $CC_1 = r$.

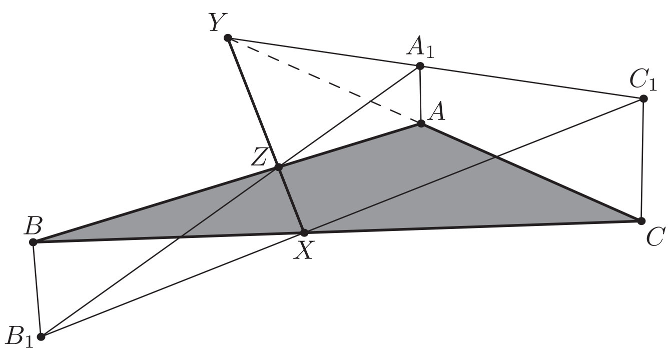

Figure 3.3B. The 3D proof of Menelaus's theorem.

One can easily check (say, by similar triangles) that the points B_1, C_1, and X are collinear. Indeed, just consider the right triangles C_1CX and B_1BX, and note the ratios of the legs. Similarly, line A_1B_1 passes through Z and A_1C_1 passes through Y.

But now consider the plane $\mathcal{Q}$ of the triangle $A_1B_1C_1$. The intersection of planes $\mathcal{P}$ and $\mathcal{Q}$ is a line. However, this line contains the points X, Y, Z, so we are done. □

It also turns out that Ceva's theorem (as well as its trigonometric form) can be generalized using directed lengths. We can write this in the following manner. This should be taken as the full form of Ceva's theorem.

Theorem 3.8 (Ceva's Theorem with Directed Lengths). *Let ABC be a triangle and X, Y, Z be points on lines BC, CA, AB distinct from its vertices. Then lines AX, BY, CZ are concurrent if and only if*

$$\frac{AZ}{ZB} \cdot \frac{BX}{XC} \cdot \frac{CY}{YA} = 1$$

where the ratios are directed.

The condition is equivalent to

$$\frac{\sin \angle BAX \sin \angle CBY \sin \angle ACZ}{\sin \angle XAC \sin \angle YBA \sin \angle ZCB} = 1$$

where either exactly one or exactly three of X, Y, Z lie strictly inside sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$. Because exactly two altitudes land outside the sides in an obtuse triangle, this generalization lets us complete the proof that the orthocenter exists for obtuse triangles. (What about for right triangles?)

3.4 The Centroid and the Medial Triangle

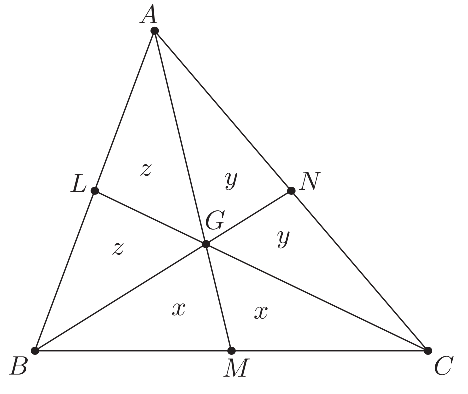

Figure 3.4A. Area ratios on the centroid of a triangle.

We can say even more about the centroid than just its existence by again considering area ratios. Consider Figure 3.4A, where we have added the midpoints of each of the sides (the triangle they determine is called the **medial triangle**). Notice that

$$1 = \frac{BM}{MC} = \frac{[GMB]}{[CMG]}$$

as discussed before in our proof of Ceva's theorem. Consequently $[GMB] = [CMG]$ and so we mark their areas with an x in Figure 3.4A. We can similarly define y and z.

But now, by the exact same reasoning,

$$1 = \frac{BM}{MC} = \frac{[AMB]}{[CMA]} = \frac{x + 2z}{x + 2y}.$$

Hence $y = z$. Analogous work gives $x = y$ and $x = z$. So that means the six areas of the triangles are all equal.

In that vein, we deduce

$$\frac{AG}{GM} = \frac{[GAB]}{[MGB]} = \frac{2z}{x} = 2.$$

This yields an important fact concerning the centroid of the triangle.

Lemma 3.9 (Centroid Division). *The centroid of a triangle divides the median into a* $2:1$ *ratio.*

Just how powerful can area ratios become? Answer: you can build a whole coordinate system around them. See Chapter 7.

3.5 Homothety and the Nine-Point Circle

First of all, what is a homothety? A **homothety** or **dilation** is a special type of similarity, in which a figure is dilated from a center. See Figure 3.5A.

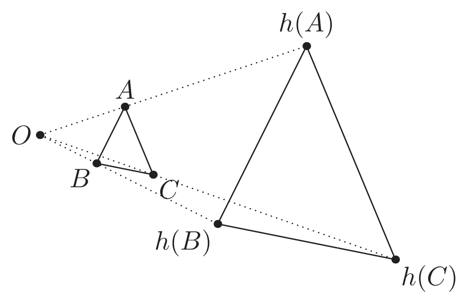

Figure 3.5A. A homothety h with center O acting on ABC.

More formally, a homothety h is a transformation defined by a center O and a real number k. It sends a point P to another point $h(P)$, multiplying the distance from O by k. The number k is the **scale factor**. It is important to note that k can be negative, in which case we have a **negative homothety**. See Figure 3.5B.

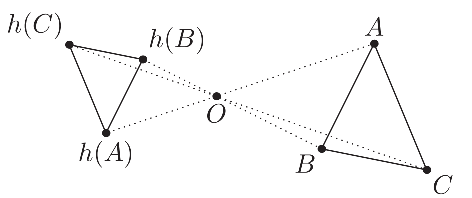

Figure 3.5B. A negative homothety with center O.

In other words, all this is a fancy special case of similar triangles.

Homothety preserves many things, including but not limited to tangency, angles (both vanilla and directed), circles, and so on. They do not preserve length, but they work well enough: the lengths are simply all multiplied by k.

Furthermore, given noncongruent parallel segments $\overline{AB}$ and $\overline{XY}$ (what happens if $AB = XY$?), we can consider the intersection point O of lines AX and BY. This is the center of a homothety sending one segment to the other. (As is the intersection of lines AY and BX—one of these is negative.) As a result, parallel lines are often indicators of homotheties.

A consequence of this is the following useful lemma.

Lemma 3.10 (Homothetic Triangles). *Let ABC and XYZ be non-congruent triangles such that $\overline{AB} \parallel \overline{XY}$, $\overline{BC} \parallel \overline{YZ}$, and $\overline{CA} \parallel \overline{ZX}$. Then lines AX, BY, CZ concur at some point O, and O is a center of a homothety mapping $\triangle ABC$ to $\triangle XYZ$.*

Convince yourself that this is true. The proof is to take a homothety h with $X = h(A)$ and $Y = h(B)$ and then check that we must have $Z = h(C)$.

One famous application of homothety is the so-called **nine-point circle**. Recall Lemma 1.17, which states that the reflection of the orthocenter over $\overline{BC}$, as well as the reflection over the midpoint of $\overline{BC}$, lies on (ABC). In Figure 3.5C, we have added in the reflections over the other sides as well.

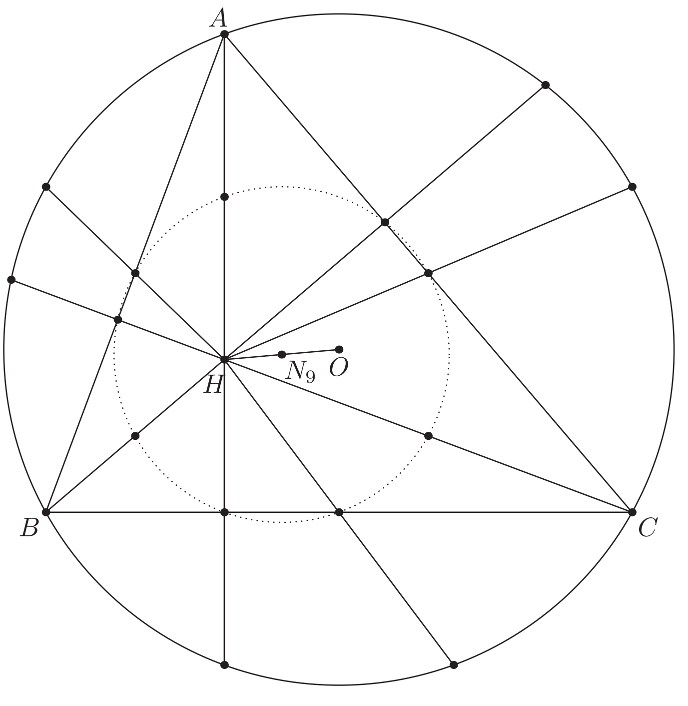

Figure 3.5C. The nine-point circle.

We now have nine points on (ABC) with center O, the three reflections of H over the sides, the three reflections of H over the midpoints, and the vertices of the triangle themselves.

Let us now take a homothety h at H (meaning with center H) and with scale factor $\frac{1}{2}$. This brings all the reflections back onto the sides of ABC, while also giving us as an added bonus the midpoints of $\overline{AH}$, $\overline{BH}$, $\overline{CH}$. In addition, O gets mapped to the midpoint of $\overline{OH}$, say N_9.

On the other hand homothety preserves circles, so astonishingly enough, these nine points remain concyclic. We even know the center of the circle—it is the image $h(O) = N_9$, called the **nine-point center**. We even know the radius! It is just half of the circumradius (ABC). This circles is called the nine-point circle.

Lemma 3.11 (Nine-Point Circle). *Let ABC be a triangle with circumcenter O and orthocenter H, and denote by N_9 the midpoint of $\overline{OH}$. Then the midpoints of $\overline{AB}$, $\overline{BC}$, $\overline{CA}$, $\overline{AH}$, $\overline{BH}$, $\overline{CH}$, as well as the feet of the altitudes of $\triangle ABC$, lie on a circle centered at N_9. Moreover, the radius of this circle is half the radius of (ABC).*

We will see several more applications of homothety in Chapter 4, but this is one of the most memorable. A second application is the **Euler line**—the circumcenter, orthocenter, and centroid are collinear as well! We leave this famous result as Lemma 3.13; see Figure 3.5D.

Problems for this Section

Problem 3.12. Give an alternative proof of Lemma 3.9 by taking a negative homothety. **Hints**: 360 165 348

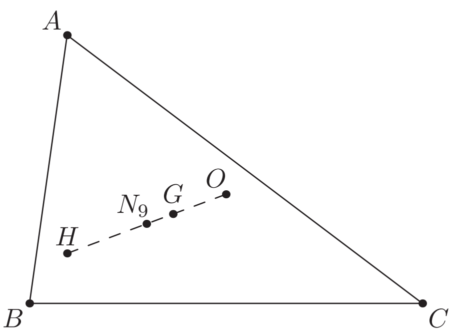

Figure 3.5D. The Euler line of a triangle.

Lemma 3.13 (Euler Line). *In triangle ABC, prove that O, G, H (with their usual meanings) are collinear and that G divides $\overline{OH}$ in a* $2:1$ *ratio.* **Hints**: 426 47 314

3.6 Example Problems

Our first example is from the very first European Girl's Math Olympiad. It is a good example of how recognizing one of our configurations (in this case, the reflections of the orthocenters) can lead to an elegant solution.

Example 3.14 (EGMO 2012/7). Let ABC be an acute-angled triangle with circumcircle Γ and orthocenter H. Let K be a point of Γ on the other side of $\overline{BC}$ from A. Let L be the reflection of K across $\overline{AB}$, and let M be the reflection of K across $\overline{BC}$. Let E be the second point of intersection of Γ with the circumcircle of triangle BLM. Show that the lines KH, EM, and BC are concurrent.

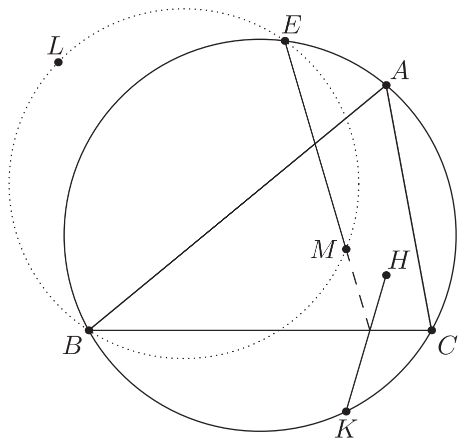

Figure 3.6A. From the first European Girl's Olympiad.

Upon first reading the problem, there are two observations we can make about it.

1. There are a lot of reflections.
2. The orthocenter does not do anything until the last sentence, when it magically appears as the endpoint of one of the concurrent lines.

This is a pretty tell-tale sign. What does the orthocenter have to do with reflections and the circumcircle? We need to tie in the orthocenter *somehow*, otherwise it is just floating in the middle of nowhere. How do we do this?

These questions motivate us to reflect H over $\overline{BC}$ and $\overline{AB}$ to points H_A and H_C, corresponding to the reflections of K across these segments. This move incorporates both the observations above. At this point we realize that $\overline{MH_A}$ and $\overline{HK}$ concur on $\overline{BC}$ for obvious reasons. So the problem is actually asking to show that H_A, M, and E are collinear. This is certainly progress.

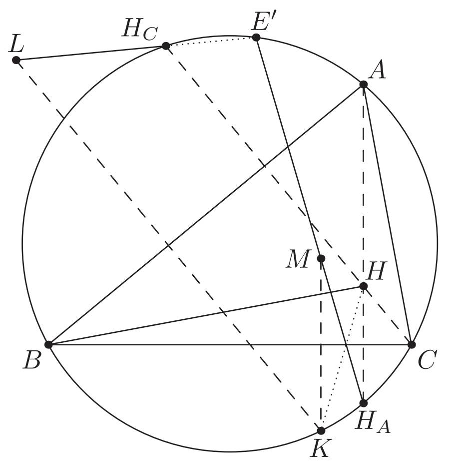

Figure 3.6B. Adding in some reflections.

At this point we can instead let E' be the intersection of H_AM with Γ and try to show that $BLE'M$ is concyclic. We are motivated to use phantom points to handle collinearity (since "concyclic" is easier to show), and we choose E because H_A and M are simpler—they are just reflections of given points. (Of course, it is probably possible to rewrite the proof without phantom points.) In any case, it suffices to prove $\measuredangle LE'M = \measuredangle LBM$.

However, we can compute $\measuredangle LBM$ easily. It is just

$$\measuredangle LBK + \measuredangle KBM = 2(\measuredangle ABK + \measuredangle KBC) = 2\measuredangle ABC.$$

So now we have reduced this to showing that $\measuredangle LE'M = 2\measuredangle ABC$.

Examining the scaled diagram closely suggests that L, H_C, and E' might be collinear. Is this true? It would sure seem so. To see how useful our conjecture might be, we quickly conjure

$$\measuredangle H_CE'H_A = \measuredangle H_CBH_A = 2\measuredangle ABC.$$

Thus the desired conclusion is actually equivalent to showing these three points are collinear. Now we certainly want to establish this.

How do we go about proving this? Angle chasing seems the most straightforward. It would suffice to prove that $\measuredangle LH_CB = \measuredangle E'H_CB$; the latter is equal to $\measuredangle E'H_AB$, which by symmetry happens to equal $\measuredangle BHK$. So we need $\measuredangle LH_CB = \measuredangle BHK$—which is clear by symmetry.

Solution to Example 3.14. Let H_A and H_C be the reflections of H across $\overline{BC}$ and $\overline{BA}$, which lie on Γ. Let E' be the second intersection of line H_AM with Γ. By construction, lines $E'M$ and HK concur on $\overline{BC}$. First, we claim that L, H_C, and E' are collinear. By reflections,

$$\measuredangle LH_CB = -\measuredangle KHB = \measuredangle MH_AB$$

and

$$\measuredangle MH_AB = \measuredangle E'H_AB = \measuredangle E'H_CB$$

as desired. Now,

$$\measuredangle LE'M = \measuredangle H_CE'H_A = \measuredangle H_CBH_A = 2\measuredangle ABC$$

and

$$\measuredangle LBM = \measuredangle LBK + \measuredangle KBM = 2\measuredangle ABK + 2\measuredangle KBC = 2\measuredangle ABC$$

so B, L, E', M are concyclic. Hence $E = E'$ and we are done. □

The second example is similar in spirit.

Example 3.15 (Shortlist 2000/G3). Let O be the circumcenter and H the orthocenter of an acute triangle ABC. Show that there exist points D, E, and F on sides BC, CA, and AB respectively such that

$$OD + DH = OE + EH = OF + FH$$

and the lines AD, BE, and CF are concurrent.

The weird part of this problem is the sum condition. Why $OD + DH = OE + EH = OF + FH$? The good news is that at least we can (try to) pick the points D, E, F. So we focus on using this to get rid of the strange condition. Are there any choices of D, E, F that readily satisfy the condition, and which induce concurrent cevians?

Having a ruler and compass is important here. After you make a guess for the points D, E, F, you better make sure that the three lines look concurrent. It is helpful to have more than one diagram for this.

One guess might be to use orthocenter reflections again. If we let H_A, H_B, H_C denote the reflections, then $OD + DH_A = OE + EH_B = OF + FH_C$. Hence we can just pick let D be the intersection of $\overline{OH_A}$ and $\overline{BC}$, and define E and F similarly. Then $OD + DH_A = OE + EH_B = OF + FH_C = R$, where R is the circumradius of $\triangle ABC$.

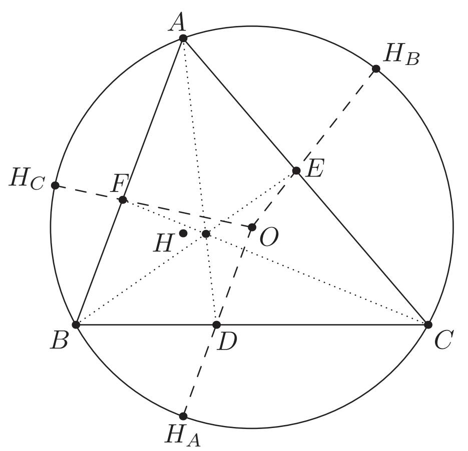

Figure 3.6C. Reflecting the orthocenter again for Example 3.15.

Now the moment of truth—are we lucky enough that the cevians concur? The computer-generated Figure 3.6C probably gives it away, but draw a diagram or two and convince yourself. This is how you check if you are going in the right direction on a contest.

Once convinced of that, we are in good shape. We just need to show that the cevians concur. Naturally, we fall back to Ceva's theorem for that. Unfortunately, we do not know much in the way of lengths (other than the carefully contrived $OD + DH = R$). Nor do we know much about the angles $\angle BAD$ and $\angle CAD$. So how else can we compute $\frac{BD}{CD}$? This is all we need, since once $\frac{BD}{CD}$ is found, we simply find the other two ratios in the same manner and multiply all three together. This product must be one, at which point we win.

The main idea now is to use the law of sines. Let us focus on triangles BDH_A and CDH_A. Because H_A was the reflection of an orthocenter, we know a lot about its angles. Specifically,

$$\measuredangle H_A BD = \measuredangle H_A BC = -\measuredangle HBC = 90^\circ - C$$

and

$$\measuredangle DH_AB = \measuredangle OH_AB = 90^\circ - \measuredangle BAH_A = 90^\circ - \measuredangle BAH = B$$

where $\measuredangle BH_AO = 90^\circ - \measuredangle BAH_A$ follows from Lemma 1.30. (Although I am mainly using directed angles here from force of habit; ABC is acute so this could likely be avoided.)

This is good, because the law of sines now lets us compute useful ratios. Noting that our angles were directed positively (that is, $\measuredangle H_ABD$ and $\measuredangle DH_AB$ both are counterclockwise), we can apply the law of sines to obtain

$$\frac{BD}{DH_A} = \frac{\sin \angle DH_AB}{\sin \angle H_ABD} = \frac{\sin B}{\cos C}.$$

The similar equation for $\triangle CDH_A$ is

$$\frac{CD}{DH_A} = \frac{\sin C}{\cos B}$$

and upon dividing we obtain

$$\frac{BD}{CD} = \frac{\sin B \cos B}{\sin C \cos C}.$$

Thus

$$\frac{CE}{EA} = \frac{\sin C \cos C}{\sin A \cos A}$$

and

$$\frac{AF}{FB} = \frac{\sin A \cos A}{\sin B \cos B}$$

and Ceva's theorem completes the solution.

A second alternative approach for obtaining the ratio $\frac{BD}{CD}$ involves the law of sines on triangle BOC. We present it below.

Solution to Example 3.15. Let H_A, H_B, H_C denote the reflections of H over $\overline{BC}$, $\overline{CA}$, $\overline{AB}$, respectively. Let D denote the intersection of $\overline{OH_A}$ and $\overline{BC}$. Evidently $OD + DH = OD + DH_A$ is the radius of (ABC). Hence if we select E and F analogously, we obtain $OD + DH = OE + EH = OF + FH$.

We now show that $\overline{AD}$, $\overline{BE}$, $\overline{CF}$ are concurrent. Let R denote the circumradius of ABC. By the law of sines on $\triangle OBD$, we find that

$$\frac{BD}{R} = \frac{\sin \angle BOD}{\sin \angle BDO} = \frac{\sin 2\angle BAH_A}{\sin \angle BDO} = \frac{\sin 2B}{\sin \angle BDO}.$$

Similarly,

$$\frac{CD}{R} = \frac{\sin 2C}{\sin \angle CDO}$$

whence dividing gives

$$\frac{BD}{CD} = \frac{\sin 2B}{\sin 2C}.$$

It follows that

$$\frac{BD}{CD} \cdot \frac{CE}{EA} \cdot \frac{BF}{FA} = 1$$

and hence we are done by Ceva's theorem. □

What is the moral of the story here? First of all, good diagrams are really important for making sure what you are trying to prove is true. Secondly, flipping the orthocenter over the sides is a useful trick (though not the only one) for floating orthocenters that do not seem connected to anything else in the diagram. Thirdly, you should think of Ceva's theorem whenever you are going after a symmetric concurrency (as in this problem), since this lets you focus on just one third of the diagram and use symmetry on the other two-thirds. And finally, when you need ratios but only have angles, you can often make the connection via the law of sines.

3.7 Problems

Problem 3.16. Let ABC be a triangle with contact triangle DEF. Prove that $\overline{AD}$, $\overline{BE}$, $\overline{CF}$ concur. The point of concurrency is the **Gergonne point**[‡] of triangle ABC. **Hint**: 683

Lemma 3.17. *In cyclic quadrilateral $ABCD$, points X and Y are the orthocenters of $\triangle ABC$ and $\triangle BCD$. Show that $AXYD$ is a parallelogram.* **Hints**: 410 238 592 **Sol:** p.246

Problem 3.18. Let $\overline{AD}$, $\overline{BE}$, $\overline{CF}$ be concurrent cevians in a triangle, meeting at P. Prove that

$$\frac{PD}{AD} + \frac{PE}{BE} + \frac{PF}{CF} = 1.$$

Hints: 339 16 46

Problem 3.19 (Shortlist 2006/G3). Let $ABCDE$ be a convex pentagon such that

$$\angle BAC = \angle CAD = \angle DAE \quad \text{and} \quad \angle ABC = \angle ACD = \angle ADE.$$

Diagonals BD and CE meet at P. Prove that ray AP bisects $\overline{CD}$. **Hints**: 31 61 478 **Sol:** p.247

Problem 3.20 (BAMO 2013/3). Let H be the orthocenter of an acute triangle ABC. Consider the circumcenters of triangles ABH, BCH, and CAH. Prove that they are the vertices of a triangle that is congruent to ABC. **Hints**: 119 200 350

Problem 3.21 (USAMO 2003/4). Let ABC be a triangle. A circle passing through A and B intersects segments AC and BC at D and E, respectively. Lines AB and DE intersect at F, while lines BD and CF intersect at M. Prove that $MF = MC$ if and only if $MB \cdot MD = MC^2$. **Hints**: 662 480 446

Theorem 3.22 (Monge's Theorem). *Consider disjoint circles ω_1, ω_2, ω_3 in the plane, no two congruent. For each pair of circles, we construct the intersection of their common external tangents. Prove that these three intersections are collinear.* **Hints**: 102 48 **Sol:** p.247

[‡] Take note: the Gergonne point is not the incenter!

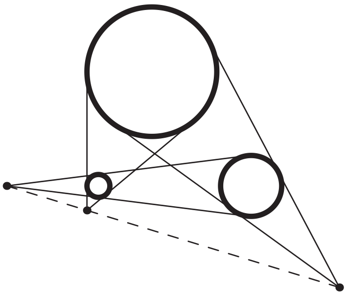

Figure 3.7A. Monge's theorem. The three points are collinear.

Theorem 3.23 (Cevian Nest). *Let $\overline{AX}$, $\overline{BY}$, $\overline{CZ}$ be concurrent cevians of ABC. Let $\overline{XD}$, $\overline{YE}$, $\overline{ZF}$ be concurrent cevians in triangle XYZ. Prove that rays AD, BE, CF concur.* **Hints**: 284 613 591 225 **Sol:** p.248

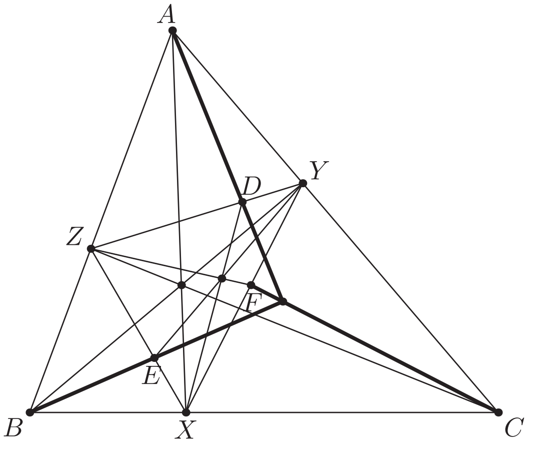

Figure 3.7B. Cevian nest

Problem 3.24. Let ABC be an acute triangle and suppose X is a point on (ABC) with $\overline{AX} \parallel \overline{BC}$ and $X \neq A$. Denote by G the centroid of triangle ABC, and by K the foot of the altitude from A to $\overline{BC}$. Prove that K, G, X are collinear. **Hints**: 671 248 244

Problem 3.25 (USAMO 1993/2). Let $ABCD$ be a quadrilateral whose diagonals $\overline{AC}$ and $\overline{BD}$ are perpendicular and intersect at E. Prove that the reflections of E across $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$ are concyclic. **Hints**: 272 491 265

Problem 3.26 (EGMO 2013/1). The side BC of the triangle ABC is extended beyond C to D so that $CD = BC$. The side CA is extended beyond A to E so that $AE = 2CA$. Prove that if $AD = BE$ then the triangle ABC is right-angled. **Hints**: 475 74 307 207 290 **Sol:** p.248

Problem 3.27 (APMO 2004/2). Let O be the circumcenter and H the orthocenter of an acute triangle ABC. Prove that the area of one of the triangles AOH, BOH, and COH is equal to the sum of the areas of the other two. **Hints**: 599 152 598 545

Problem 3.28 (Shortlist 2001/G1). Let A_1 be the center of the square inscribed in acute triangle ABC with two vertices of the square on side BC. Thus one of the two remaining vertices of the square is on side AB and the other is on AC. Points B_1 and C_1 are defined in a similar way for inscribed squares with two vertices on sides AC and AB, respectively. Prove that lines AA_1, BB_1, CC_1 are concurrent. **Hints**: 618 665 383

Problem 3.29 (USA TSTST 2011/4). Acute triangle ABC is inscribed in circle ω. Let H and O denote its orthocenter and circumcenter, respectively. Let M and N be the midpoints of sides AB and AC, respectively. Rays MH and NH meet ω at P and Q, respectively. Lines MN and PQ meet at R. Prove that $\overline{OA} \perp \overline{RA}$. **Hints**: 459 570 148 **Sol:** p.249

Problem 3.30 (USAMO 2015/2). Quadrilateral $APBQ$ is inscribed in circle ω with $\angle P = \angle Q = 90^\circ$ and $AP = AQ < BP$. Let X be a variable point on segment $\overline{PQ}$. Line AX meets ω again at S (other than A). Point T lies on arc AQB of ω such that $\overline{XT}$ is perpendicular to $\overline{AX}$. Let M denote the midpoint of chord $\overline{ST}$. As X varies on segment $\overline{PQ}$, show that M moves along a circle. **Hints**: 533 501 116 639 418

CHAPTER 4

Assorted Configurations

There is light at the end of the tunnel, but it is moving away at speed c.

There are two ways to think about the configurations in this chapter. One is as a list of configurations to be memorized and recognized on contests. Another is as just a set of problems that frequently appear as subproblems (or superproblems) of olympiad problems. We prefer the second view, and have arranged this chapter accordingly.

4.1 Simson Lines Revisited

Let ABC be a triangle and P be any point, and denote by X, Y, Z the feet of the perpendiculars from P onto lines BC, CA, and AB. From Lemma 1.48 the points X, Y, Z are collinear if and only if P lies on (ABC). When P does lie on (ABC), this is called the **Simson line** of P with respect to ABC. We can say much more about this.

Denote by H the orthocenter of triangle ABC and let line PX meet (ABC) again at a point K, and let line AH intersect the Simson line at the point L. The completed figure is shown in Figure 4.1A.

We make a few synthetic observations.

Proposition 4.1. *Prove that the Simson line is parallel to $\overline{AK}$ in the notation of Figure 4.1A.* **Hints**: 390 151

Of course $\overline{XK} \parallel \overline{AL}$, and hence we discover $LAKX$ is a parallelogram.

Problem 4.2. Let K' be the reflection of K across $\overline{BC}$. Show that K' is the orthocenter of $\triangle PBC$. **Hint**: 521

We can now apply Lemma 3.17 to deduce that $AHPK'$ is a parallelogram. Using this, one can solve the next problem.

Problem 4.3. Show that $LHXP$ is a parallelogram. **Hint**: 97

From the above we can immediately deduce Lemma 4.4.

Lemma 4.4 (Simson Line Bisection). *Let ABC be a triangle with orthocenter H. If P is a point on (ABC) then its Simson line bisects $\overline{PH}$.*

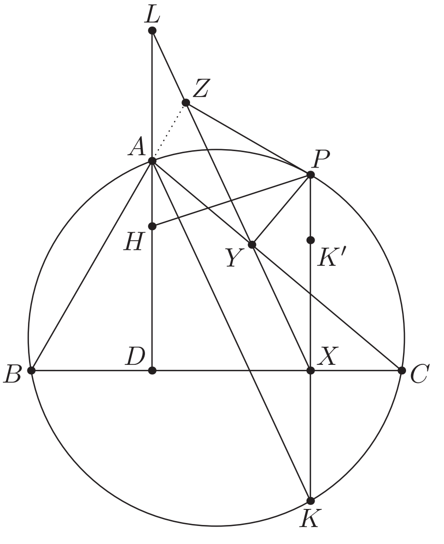

Figure 4.1A. Simson lines revisited.

Do not miss Simson lines when they appear. Contest problems that involve the Simson line usually only drop two of the altitudes and thus clandestinely construct the Simson line. Do not be fooled!

4.2 Incircles and Excircles

In Figure 4.2A we have drawn all three excenters of triangle ABC. Angle chasing gives an easy observation.

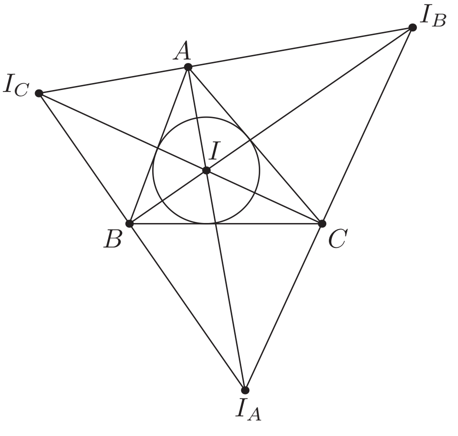

Figure 4.2A. The excenters of a triangle.

Problem 4.5. Check $\angle IAI_B = 90^\circ$ and $\angle IAI_C = 90^\circ$.

As a corollary, A lies on $\overline{I_BI_C}$. We also know (say, from Section 2.6) that the points A, I, and I_A are collinear. Actually $\overline{AI_A} \perp \overline{I_BI_C}$. Our observations can be summarized as follows.

Lemma 4.6 (Duality of Orthocenters and Excenters). *If I_A, I_B, I_C are the excenters of $\triangle ABC$, then triangle ABC is the orthic triangle of $\triangle I_A I_B I_C$, and the orthocenter is I.*

This duality is important to remember. The orthic triangle and excenters are "dual" concepts—they correspond exactly to each other. Problem writers sometimes phrase a problem stated more naturally in one framework with the other in an effort to make the problem artificially harder. Watch for this when it happens.

Problem 4.7. How are Lemma 1.18, Lemma 3.11, and Lemma 4.6 related? **Hint**: 458

Let us now concentrate further on a smaller part of the diagram. In Figure 4.2B we focus on just the A-excircle, tangent to $\overline{BC}$ at point X. We have drawn a line parallel to $\overline{BC}$ tangent again to the incircle at a point E. Suppose it intersects $\overline{AB}$ and $\overline{AC}$ at B' and C'. Evidently $\triangle AB'C'$ and $\triangle ABC$ are homothetic. But the incircle of $\triangle ABC$ is the A-excircle of $\triangle AB'C'$.

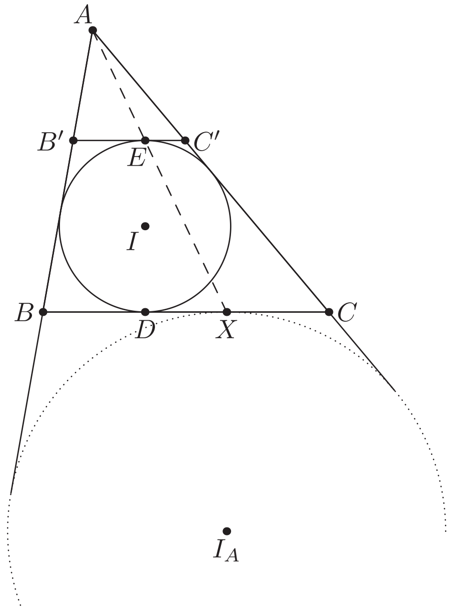

Figure 4.2B. The homothety between the incircle and A-excircle.

Problem 4.8. Prove that A, E, and X are collinear and that $\overline{DE}$ is a diameter of the incircle. **Hint**: 508

We also know that $BD = CX$, so we can actually phrase this statement without referring to the excircle.

Lemma 4.9 (The Diameter of the Incircle). *Let ABC be a triangle whose incircle is tangent to $\overline{BC}$ at D. If $\overline{DE}$ is a diameter of the incircle and ray AE meets $\overline{BC}$ at X, then $BD = CX$ and X is the tangency point of the A-excircle to $\overline{BC}$.*

Incircles and excircles often have dual properties. For example, check that the following is true as well.

Lemma 4.10 (Diameter of the Excircle). *In the notation of Lemma 4.9, suppose $\overline{XY}$ is a diameter of the A-excircle. Show that D lies on $\overline{AY}$.*

Problem for this Section

Problem 4.11. If M is the midpoint of $\overline{BC}$, prove that $\overline{AE} \parallel \overline{IM}$.

4.3 Midpoints of Altitudes

The results from the previous configuration extend to our next one. In Figure 4.3A we have removed the points B' and C' from Figure 4.2B and added an altitude $\overline{AK}$ with midpoint M. By Lemma 4.9 and Lemma 4.10, we already know that A, E, and X are collinear, as are A, D, and Y.

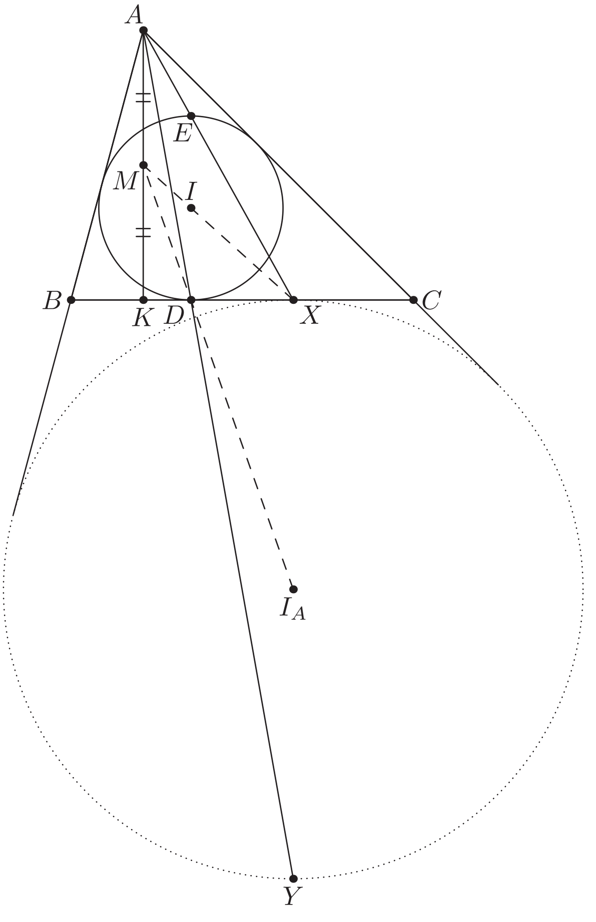

Figure 4.3A. Midpoints of altitudes.

Problem 4.12. Prove that points X, I, M are collinear. **Hints**: 138 175

Problem 4.13. Show that D, I_A, M are collinear. **Hint**: 336

We can restate these results as the following lemma.

Lemma 4.14 (Midpoint of Altitudes). *Let ABC be a triangle with incenter I and A-excenter I_A, and let D and X be the associated tangency points on $\overline{BC}$. Then lines DI_A and XI concur at the midpoint of the altitude from A.*

4.4 Even More Incircle and Incenter Configurations

Let DEF be the contact triangle of a triangle ABC, and consider the point X on $\overline{EF}$ such that $\overline{XD} \perp \overline{BC}$. The situation is shown in Figure 4.4A. The claim is that ray AX bisects $\overline{BC}$.

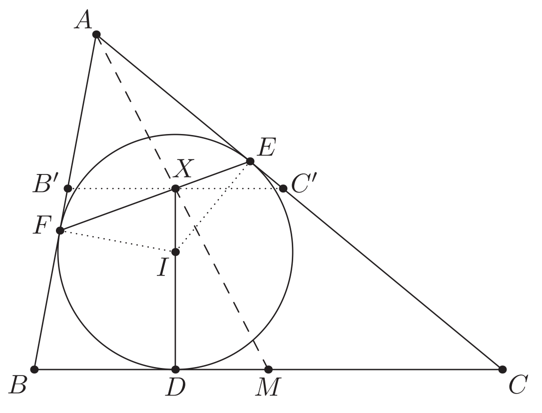

Figure 4.4A. The median intersects a side of the contact triangle.

Suppose we were trying to prove this. The key insight is that point M is kind of a distraction. We can eliminate it, along with $\overline{BC}$, by taking the line through X parallel to $\overline{BC}$ and considering a homothety. Let the line meet $\overline{AB}$ and $\overline{AC}$ again at B' and C'. Now it suffices to prove that X is the midpoint of $\overline{B'C'}$.

Problem 4.15. Show that I must lie on $(AB'C')$. **Hint**: 64

Problem 4.16. Prove that $XB' = XC'$. **Hint**: 470

Once we have these results, our next configuration is immediate.

Lemma 4.17 (An Incircle Concurrency). *Let ABC be a triangle with incenter I and contact triangle DEF. If M is the midpoint of $\overline{BC}$, then $\overline{EF}$, $\overline{AM}$ and ray DI concur.*

4.5 Isogonal and Isotomic Conjugates

This particular configuration is fairly straightforward.

Lemma 4.18 (Isogonal Conjugates). *Let ABC be a triangle and P any point not collinear with any of the sides. There exists a unique point P^* satisfying the relations*

$$\measuredangle BAP = \measuredangle P^*AC, \quad \measuredangle CBP = \measuredangle P^*BA, \quad \measuredangle ACP = \measuredangle P^*CB.$$

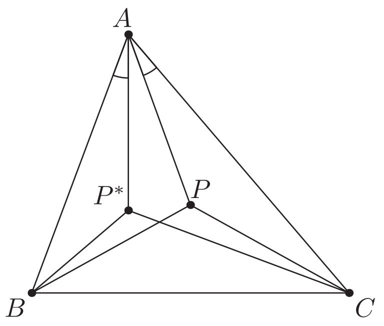

Figure 4.5A. P and P^* are isogonal conjugates.

The point P^* is called the **isogonal conjugate** of the point P. We also say line AP^* is **isogonal** to (or "is the isogonal of") line AP with respect to triangle ABC; however we often omit the phrase "with respect to triangle ABC" if the context is clear. In other words, two lines through A are isogonal if they are reflections over the angle bisector of $\angle A$.

A better way to phrase the lemma is the "buy two get one free" perspective, as in the exercise below.

Problem 4.19. Show that if two of the angle relations in Lemma 4.18 hold, then so does the third. **Hint**: 9

The **isotomic conjugate** is defined similarly. For a point P and triangle ABC, let X, Y, Z be the feet of the cevians through P. Let X' be the reflection of X about the midpoint of $\overline{BC}$ and define Y' and Z' similarly. Then the cevians $\overline{AX'}$, $\overline{BY'}$, and $\overline{CZ'}$ concur at a point P^t, the isotomic conjugate of P.

Problem 4.20. Prove that the cevians AX', BY', and CZ' concur as described above.

Problems for this Section

Problem 4.21. Check that if Q is the isogonal conjugate of P, then P is the isogonal conjugate of Q.

Theorem 4.22 (Isogonal Ratios). *Let D and E be points on $\overline{BC}$ so that $\overline{AD}$ and $\overline{AE}$ are isogonal. Then*

$$\frac{BD}{DC} \cdot \frac{BE}{EC} = \left(\frac{AB}{AC}\right)^2.$$

Hint: 184

Problem 4.23. What is the isogonal conjugate of a triangle's circumcenter?

4.6 Symmedians

The isogonal of a median in a triangle is called a **symmedian**. The concurrency point of the three symmedians is the isogonal conjugate of the centroid, called the **symmedian point**.

Symmedians have tons of nice properties. We first show how they arise naturally.

Lemma 4.24 (Constructing the Symmedian). *Let X be the intersection of the tangents to (ABC) at B and C. Then line AX is a symmedian.*

The proof is a direct computation with the law of sines. Let M be the intersection of the isogonal of AX on $\overline{BC}$; we wish to prove that M is the midpoint of $\overline{BC}$.

Problem 4.25. Show that

$$\frac{BM}{MC} = \frac{\sin\angle B \sin\angle BAX}{\sin\angle C \sin\angle CAX} = 1.$$

Now let us describe several additional properties of symmedians.

Lemma 4.26 (Properties of the Symmedian). *Let ABC be a triangle, and let the tangents to its circumcircle at B and C meet at point X. Let $\overline{AX}$ meet (ABC) again at K and $\overline{BC}$ at D. Then $\overline{AD}$ is the A-symmedian and*

(a) $\overline{KA}$ is a K-symmedian of $\triangle KBC$.
(b) $\triangle ABK$ and $\triangle AMC$ are directly similar.
(c) We have

$$\frac{BD}{DC} = \left(\frac{AB}{AC}\right)^2.$$

(d) We have

$$\frac{AB}{BK} = \frac{AC}{CK}.$$

(e) (BCX) passes through the midpoint of $\overline{AK}$.
(f) $\overline{BC}$ is the B-symmedian of $\triangle BAK$, and the C-symmedian of $\triangle CAK$.
(g) $\overline{BC}$ is the interior angle bisector of $\angle AMK$, and $\overline{MX}$ is the exterior angle bisector.

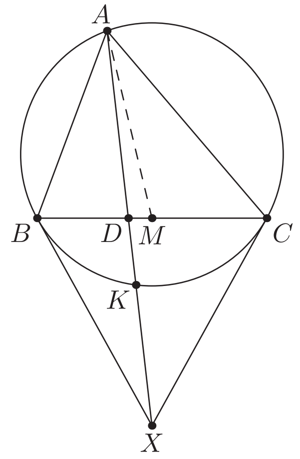

Figure 4.6A. The A-symmedian of a triangle

Here property (a) is obvious from the tangent construction, while (c) is a special case of Theorem 4.22. Properties (b) and (e) follow from straightforward angle chasing. The rest of the properties are described in the exercises. Extracting some of these properties yields the following.

Lemma 4.27 (Symmedians in Cyclic Quadrilaterals). *Let $ABCD$ be a cyclic quadrilateral. The following are equivalent.*

(a) $AB \cdot CD = BC \cdot DA$.
(b) $\overline{AC}$ *is an* A*-symmedian of* $\triangle DAB$.
(c) $\overline{AC}$ *is a* C*-symmedian of* $\triangle BCD$.
(d) $\overline{BD}$ *is a* B*-symmedian of* $\triangle ABC$.
(e) $\overline{BD}$ *is a* D*-symmedian of* $\triangle CDA$.

In Chapter 9, we learn that such a quadrilateral is called a **harmonic quadrilateral**, and possesses even more interesting properties.

Problems for this Section

Problem 4.28. Verify (d) of Lemma 4.26. **Hint**: 194

Problem 4.29. Show that (f) of Lemma 4.26 follows (with some effort) from (d). **Hints**: 190 628 584

Problem 4.30. Prove (g) of Lemma 4.26. **Hints**: 65 474

4.7 Circles Inscribed in Segments

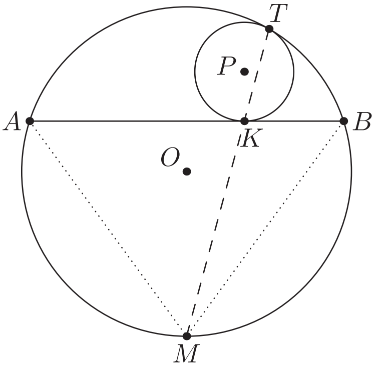

Figure 4.7A. A circle is inscribed in a segment.

Our next configuration involves a tangent circle. Let Ω be a circle with center O and a chord $\overline{AB}$, and consider a circle ω tangent internally to Ω at T and to $\overline{AB}$ at K. Let M denote the midpoint of the arc $\widehat{AB}$ not containing T. For no good reason, the region bounded by $\overline{AB}$ and the other arc $\widehat{AB}$ containing T is called a **segment**, hence the title of this section.

As the centers of ω and Ω are collinear with T (by tangency), it follows there is a homothety at T mapping ω to Ω.

Problem 4.31. Show that this homothety takes K to M, and in particular that T, K, and M are collinear.

Problem 4.32. Show that $\triangle TMB \sim \triangle BMK$.

The last implication gives that $MK \cdot MT = MB^2$. So, we deduce the following.

Lemma 4.33 (Circles Inscribed in Segments). *Let $\overline{AB}$ be a chord of a circle Ω. Let ω be a circle tangent to chord $\overline{AB}$ at K and internally tangent to ω at T. Then ray TK passes through the midpoint M of the arc $\widehat{AB}$ not containing T.*

Moreover, $MA^2 = MB^2$ is the power of M with respect to ω.

This configuration is even more straightforward with inversion, discussed in Chapter 8. A reader comfortable with inversion is encouraged to reconstruct the proof using a suitable inversion at M.

The above configuration extends naturally to the next one, shown in Figure 4.7B. Let C be another point on arc $\widehat{AB}$ containing T, and let D be a point on $\overline{AB}$ such that $\overline{CD}$ is tangent to ω at L.

The circle ω is called a **curvilinear incircle** of ABC. (As D varies along $\overline{AB}$, we obtain many curvilinear incircles, hence we refer to "a" curvilinear incircle. The next section discusses the special case $A = D$.) We claim that if I is the intersection of $\overline{CM}$ and $\overline{KL}$, then I is the incenter of $\triangle ABC$.

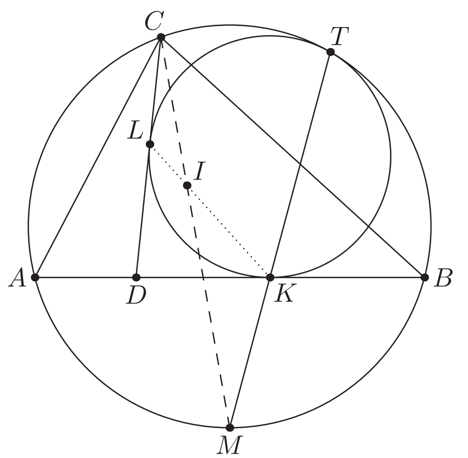

Figure 4.7B. More unusual tangent circles.

Problem 4.34. Prove that the points C, L, I, T are concyclic. **Hints**: 69 273 140

Problem 4.35. Show that $\triangle MKI \sim \triangle MIT$, and that the triangles are oppositely oriented. **Hints**: 472 236

Finally, how do we derive that I is the incenter? The similarity above gives that $MI^2 = MK \cdot MT$, but yet

$$MK \cdot MT = MA^2 = MB^2$$

by Lemma 4.33. Hence $MI = MA = MB$, and Lemma 1.18 establishes the configuration below.

Lemma 4.36 (Curvilinear Incircle Chords). *Let ABC be a triangle and D be a point on $\overline{AB}$. Suppose a circle ω is tangent to $\overline{CD}$ at L, $\overline{AB}$ at K, and also to (ABC). Then the incenter of ABC lies on line LK.*

4.8 Mixtilinear Incircles

The ***A*-mixtilinear incircle** of a triangle ABC is the circle internally tangent to (ABC), as well as to sides $\overline{AB}$ and $\overline{AC}$.

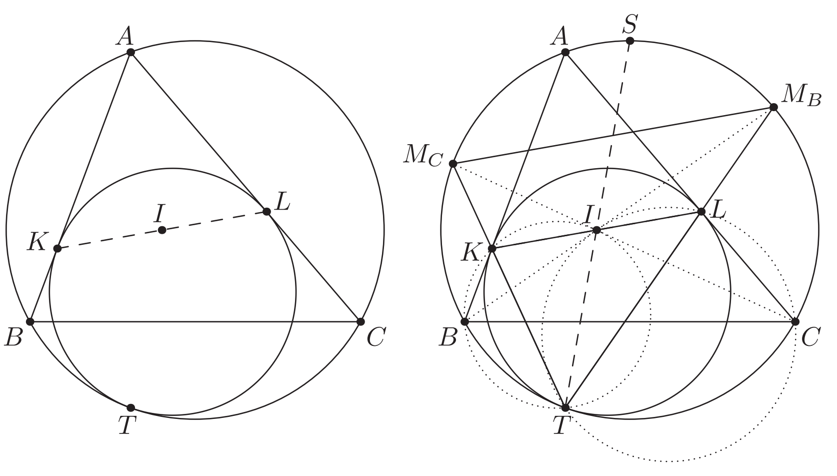

Figure 4.8A. An A-mixtilinear incircle.

Throughout this section, we let ω_A refer to this A-mixtilinear circle. Let T denote the tangency point of the ω_A with (ABC), and K and L the tangency points on $\overline{AB}$ and $\overline{AC}$. Taking $D = A$ in Lemma 4.36, we know that the incenter I of $\triangle ABC$ lies on $\overline{KL}$.

Problem 4.37. Using the fact that I lies on $\overline{KL}$, check that I is in fact the midpoint of $\overline{KL}$.

In Chapter 9 we give a nice alternative proof that I is the midpoint of $\overline{KL}$ using Pascal's theorem.

Let us see if we can learn anything interesting about the point T now. Let M_C and M_B be the midpoints of arcs $\widehat{AB}$ and $\widehat{AC}$. We of course already know (from Lemma 4.33) that T is the intersection of lines KM_C and LM_B. Now, extend line TI to meet the circumcircle of $\triangle ABC$ again at point S. The completed figure is show in Figure 4.8A.

Problem 4.38. Prove that $\angle ATK = \angle LTI$. **Hint**: 469

Problem 4.39. Prove that S is the midpoint of the arc $\widehat{BC}$ containing A. **Hint**: 342

Hence, we deduce that line TI passes through the midpoint of arc $\widehat{BC}$ not containing T. A second way to prove this is through angle chasing: one can show* that quadrilaterals $BKIT$ and $CLIT$ are cyclic since

$$\measuredangle IKT = \measuredangle LKT = \measuredangle M_B M_C T = \measuredangle M_B BT = \measuredangle IBT.$$

In any case this gives us $\measuredangle M_C TS = \measuredangle KTI = \measuredangle KBI = \measuredangle ABI$ for free, allowing us to establish the same conclusion as before.

In Chapter 8, we also prove (as part of Problem 8.31) that if E is the contact point of the A-excircle with $\overline{BC}$, then $\overline{AT}$ and $\overline{AE}$ are isogonal. Moreover, as Problem 4.49 we ask the reader to prove that the isogonal of $\overline{TA}$ with respect to $\triangle TBC$ passes through the contact point of the incircle at $\overline{BC}$. These additional results are exhibited in Figure 4.8B.

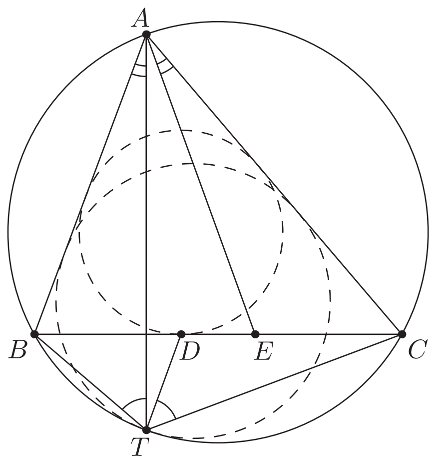

Figure 4.8B. Segments $\overline{AT}$ and $\overline{AE}$ are isogonal in $\triangle ABC$, while segments $\overline{TD}$ and $\overline{TA}$ are isogonal in $\triangle TBC$.

Combining the results in Figure 4.8A and Figure 4.8B into one big lemma:

Lemma 4.40 (Mixtilinear Incircles). *Let ABC be a triangle and let its A-mixtilinear circle be tangent to $\overline{AB}$, $\overline{AC}$, and (ABC) at K, L, and T, respectively. Denote by D and E the contact points of the incircle and A-excircle on $\overline{BC}$, respectively.*

(a) The midpoint I of $\overline{KL}$ is the incenter of $\triangle ABC$.
(b) Lines TK and TL pass through the midpoints of arcs $\widehat{AB}$ and $\widehat{AC}$ not containing T.
(c) Line TI passes through the midpoint of arc $\widehat{BC}$ containing A.
(d) The angles $\angle BAT$ and $\angle CAE$ are equal.
(e) The angles $\angle BTA$ and $\angle CTD$ are equal.
(f) Quadrilaterals $BKIT$ and $CLIT$ are concyclic.

For even more, see Lemma 7.42.

* Actually, we already proved this during our proof of Lemma 4.36.

4.9 Problems

These are not in any order—I cannot spoil the fun here!

Problem 4.41 (Hong Kong 1998). Let $PQRS$ be a cyclic quadrilateral with $\angle PSR = 90^\circ$ and let H and K be the feet of the altitudes from Q to lines PR and PS. Prove that $\overline{HK}$ bisects $\overline{QS}$. **Hints**: 267 420

Problem 4.42 (USAMO 1988/4). Suppose $\triangle ABC$ is a triangle with incenter I. Show that the circumcenters of $\triangle IAB$, $\triangle IBC$, and $\triangle ICA$ lie on a circle whose center is the circumcenter of $\triangle ABC$. **Hint**: 249 **Sol:** p.249

Problem 4.43 (USAMO 1995/3). Given a nonisosceles, nonright triangle ABC, let O denote its circumcenter, and let A_1, B_1, and C_1 be the midpoints of sides $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, respectively. Point A_2 is located on the ray OA_1 so that $\triangle OAA_1$ is similar to $\triangle OA_2A$. Points B_2 and C_2 on rays OB_1 and OC_1, respectively, are defined similarly. Prove that lines AA_2, BB_2, and CC_2 are concurrent. **Hints**: 691 550 128

Problem 4.44 (USA TST 2014). Let ABC be an acute triangle and let X be a variable interior point on the minor arc $\widehat{BC}$. Let P and Q be the feet of the perpendiculars from X to lines CA and CB, respectively. Let R be the intersection of line PQ and the perpendicular from B to $\overline{AC}$. Let ℓ be the line through P parallel to $\overline{XR}$. Prove that as X varies along minor arc $\widehat{BC}$, the line ℓ always passes through a fixed point. **Hints**: 45 424 **Sol:** p.249

Problem 4.45 (USA TST 2011/1). In an acute scalene triangle ABC, points D, E, F lie on sides BC, CA, AB, respectively, such that $\overline{AD} \perp \overline{BC}$, $\overline{BE} \perp \overline{CA}$, $\overline{CF} \perp \overline{AB}$. Altitudes $\overline{AD}, \overline{BE}, \overline{CF}$ meet at orthocenter H. Points P and Q lie on segment $\overline{EF}$ such that $\overline{AP} \perp \overline{EF}$ and $\overline{HQ} \perp \overline{EF}$. Lines DP and QH intersect at point R. Compute HQ/HR. **Hints**: 124 317 26 **Sol:** p.250

Problem 4.46 (ELMO Shortlist 2012). Circles Ω and ω are internally tangent at point C. Chord AB of Ω is tangent to ω at E, where E is the midpoint of $\overline{AB}$. Another circle, ω_1 is tangent to Ω, ω, and $\overline{AB}$ at D, Z, and F respectively. Rays CD and AB meet at P. If $M \neq C$ is the midpoint of major arc AB, show that

$$\tan \angle ZEP = \frac{PE}{CM}.$$

Hints: 370 40 672 211

Problem 4.47 (USAMO 2011/5). Let P be a point inside convex quadrilateral $ABCD$. Points Q_1 and Q_2 are located within $ABCD$ such that

$$\angle Q_1BC = \angle ABP, \quad \angle Q_1CB = \angle DCP,$$
$$\angle Q_2AD = \angle BAP, \quad \angle Q_2DA = \angle CDP.$$

Prove that $\overline{Q_1Q_2} \parallel \overline{AB}$ if and only if $\overline{Q_1Q_2} \parallel \overline{CD}$. **Hints**: 4 528

Problem 4.48 (Japanese Olympiad 2009). Triangle ABC is inscribed in circle Γ. A circle with center O is drawn, tangent to side BC at a point P, and internally tangent to the arc BC

of Γ not containing A at a point Q. Show that if $\angle BAO = \angle CAO$ then $\angle PAO = \angle QAO$.
Hints: 220 676 19

Problem 4.49. Let ABC be a triangle and let its incircle touch $\overline{BC}$ at D. Let T be the tangency point of the A-mixtilinear incircle with (ABC). Prove that $\angle BTA = \angle CTD$.
Hints: 646 529 192 425

Problem 4.50 (Vietnam TST 2003/2). Let ABC be a scalene triangle with circumcenter O and incenter I. Let H, K, L be the feet of the altitudes of triangle ABC from the vertices A, B, C, respectively. Denote by A_0, B_0, C_0 the midpoints of these altitudes $\overline{AH}$, $\overline{BK}$, $\overline{CL}$, respectively. The incircle of triangle ABC touches the sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ at the points D, E, F, respectively. Prove that the four lines A_0D, B_0E, C_0F, and OI are concurrent. **Hints**: 442 11 514 **Sol:** p.250

Problem 4.51 (Sharygin 2013). The incircle of $\triangle ABC$ touches $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ at points A', B' and C' respectively. The perpendicular from the incenter I to the C-median meets the line $A'B'$ in point K. Prove that $\overline{CK} \parallel \overline{AB}$. **Hints**: 274 551 258

Problem 4.52 (APMO 2012/4). Let ABC be an acute triangle. Denote by D the foot of the perpendicular line drawn from the point A to the side BC, by M the midpoint of $\overline{BC}$, and by H the orthocenter of ABC. Let E be the point of intersection of the circumcircle Γ of the triangle ABC and the ray MH, and F be the point of intersection (other than E) of the line ED and the circle Γ. Prove that $\frac{BF}{CF} = \frac{AB}{AC}$ must hold. **Hints**: 593 454 28 228 **Sol:** p.251

Problem 4.53 (Shortlist 2002/G7). The incircle Ω of the acute triangle ABC is tangent to $\overline{BC}$ at a point K. Let $\overline{AD}$ be an altitude of triangle ABC, and let M be the midpoint of the segment $\overline{AD}$. If N is the common point of the circle Ω and the line KM (distinct from K), then prove that the incircle Ω and the circumcircle of triangle BCN are tangent to each other at the point N **Hints**: 205 634 450 177 276

For a real challenge, check out Problem 11.19.

Part II
Analytic Techniques

CHAPTER 5

Computational Geometry

> Since you are now studying geometry and trigonometry, I will give you a problem. A ship sails the ocean. It left Boston with a cargo of wool. It grosses 200 tons. It is bound for Le Havre. The mainmast is broken, the cabin boy is on deck, there are 12 passengers aboard, the wind is blowing east-north-east, the clock points to a quarter past three in the afternoon. It is the month of May. How old is the captain?
>
> Gustave Flaubert

Suppose you are given a triangle with side lengths 13, 14, 15. Can you compute its circumradius? How about its inradius?

Up until now we have used tools from classical Euclidean geometry to develop elegant results. The following three chapters focus much more on computation: using messier methods to achieve results directly.

This chapter lays the foundation for future chapters by presenting fundamental relations between the quantities of a triangle. We also introduce Cartesian coordinates and trigonometric computation, which are capable of solving problems in their own right.

5.1 Cartesian Coordinates

The xy-plane provides a framework in which we can intersect lines, drop perpendiculars, and so on.

Unfortunately, as Cartesian coordinates are well-known to most competitors, olympiads tend to avoid problems that can be easily solved by coordinates. Because of this, we will not go into a deep exploration of their use. However, we mention one or two tricks that are less frequently seen, in the hopes that they may be helpful in a solution using Cartesian coordinates.

First is the so-called **shoelace formula**. It involves a determinant; if you are unfamiliar with determinants, consult Appendix A.1.

Theorem 5.1 (Shoelace Formula). *Consider three points $A = (x_1, y_1)$, $B = (x_2, y_2)$, and $C = (x_3, y_3)$. The signed area of triangle ABC is given by the determinant*

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

In the shoelace formula, we have used the convention of a **signed areas**. That means the area of a triangle ABC is considered positive if A, B, C appear in counterclockwise order, and negative otherwise.

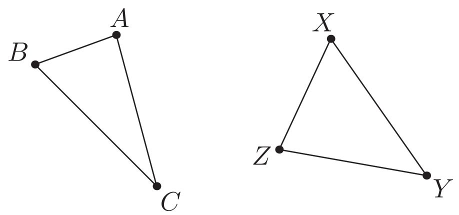

Figure 5.1A. On the left, ABC has positive signed area because its vertices are labelled counterclockwise. On the right, XYZ has negative signed area since its vertices are labelled clockwise.

The most useful special case of the shoelace formula is the following: three points are collinear if and only if the area of the "triangle" they determine is zero. Hence the shoelace formula can be used to establish collinearity. Because we are using determinants, the formula is now symmetric. The more well-known routine to establish collinearity is to verify that

$$\frac{y_3 - y_1}{x_3 - x_1} = \frac{y_2 - y_1}{x_2 - x_1},$$

which unnecessarily loses symmetry.

A second occasionally useful trick, which we state without proof:

Proposition 5.2 (Point-Line Distance Formula). *Let ℓ be the line determined by the equation $Ax + By + C = 0$. The distance from a point $P = (x_1, y_1)$ to ℓ is*

$$\frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}.$$

This allows us to compute distances from points to lines without explicitly finding the coordinates of the perpendicular foot.

Cartesian coordinates have some shortcomings, since they rely heavily on a central right angle, and there is no natural symmetric way to select the coordinates of an arbitrary triangle. Problems that can be solved by Cartesian coordinates can often also be solved more easily by complex numbers or barycentric coordinates (discussed in the next two chapters).

Put in a more positive way, problems for which coordinates are effective tend to have some defining characteristics. For example,

- The problem features a prominent right angle which can be situated at the origin, or
- The problem involves intersections or perpendiculars.

5.2 Areas

Let us now answer the question posed at the very beginning of this chapter. It turns out that one can link many important quantities of a triangle through its area.

Theorem 5.3 (Area Formulas). *The area of a triangle ABC is equal to each of the following.*

$$\begin{aligned}[ABC] &= \frac{1}{2}ab\sin C = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B \\ &= \frac{a^2\sin B\sin C}{2\sin A} \\ &= \frac{abc}{4R} \\ &= sr \\ &= \sqrt{s(s-a)(s-b)(s-c)}.\end{aligned}$$

Here $s = \frac{1}{2}(a+b+c)$ is the **semiperimeter** of the triangle, and R and r are the circumradius and inradius of $\triangle ABC$, respectively. The formula $\sqrt{s(s-a)(s-b)(s-c)}$ is often called **Heron's formula**. It has the nice property that given a, b, c, one can use it to extract r and R.

Proof. First, we establish the formula $[ABC] = \frac{1}{2}ab\sin C$ (the other formulas follow analogously). Seeing the sine, we decide to drop altitudes. Let X be the foot of the altitude from A onto $\overline{BC}$ as in Figure 5.2A, so that $[ABC] = \frac{1}{2}AX \cdot BC = \frac{1}{2}a \cdot AX$. Now observe that $AX = AC\sin C = b\sin C$ (regardless of whether $\angle C$ is acute) and hence we obtain $[ABC] = \frac{1}{2}ab\sin C$.

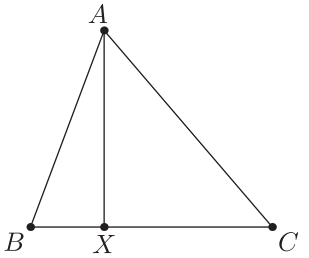

Figure 5.2A. We obtain $[ABC] = \frac{1}{2}AX \cdot BC = \frac{1}{2}ab\sin C$. This is configuration independent.

The next two lines follow from applying the extended law of sines to eliminate b or $\sin C$, respectively. Explicitly,

$$\frac{1}{2}ab\sin C = \frac{1}{2}a\left(\frac{a\sin B}{\sin A}\right)\sin C = \frac{a^2\sin B\sin C}{2\sin A}$$

and

$$\frac{1}{2}ab\sin C = \frac{1}{2}ab\left(\frac{c}{2R}\right) = \frac{abc}{4R}.$$

The proof that $[ABC] = sr$ is a cute exercise, which we leave to the reader as Problem 5.5.

Now for the least obvious step, the proof of Heron's formula. We present a proof using the following trigonometric fact.

> If x, y, z satisfy $x+y+z=180^\circ$ and $0^\circ < x, y, z < 90^\circ$, then $\tan x + \tan y + \tan z = \tan x \tan y \tan z$.

We prove this in greater generality as Proposition 6.39. Construct the contact triangle* DEF of ABC, as in Figure 5.2B.

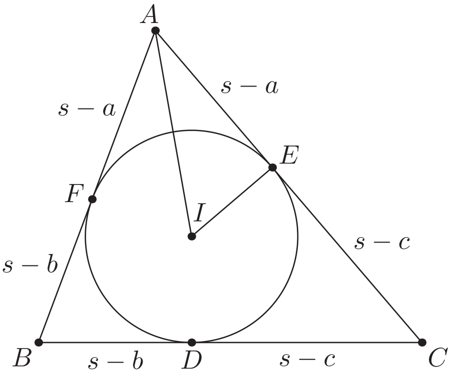

Figure 5.2B. Using the contact triangle to obtain Heron's formula.

Applying Lemma 2.15 we may deduce

$$\tan\left(90^\circ - \frac{1}{2}A\right) = \tan(\angle AIE) = \frac{s-a}{r}.$$

Similarly,

$$\tan\left(90^\circ - \frac{1}{2}B\right) = \frac{s-b}{r}$$
$$\tan\left(90^\circ - \frac{1}{2}C\right) = \frac{s-c}{r}.$$

The aforementioned trigonometric identity applies (since $270^\circ - \frac{1}{2}(A+B+C) = 180^\circ$) and yields

$$\begin{aligned}\frac{s-a}{r}\cdot\frac{s-b}{r}\cdot\frac{s-c}{r} &= \frac{s-a}{r}+\frac{s-b}{r}+\frac{s-c}{r}\\ &= \frac{3s-(a+b+c)}{r}\\ &= \frac{s}{r}.\end{aligned}$$

This gives $(sr)^2 = s(s-a)(s-b)(s-c)$ as desired. $\square$

We can now answer the question posed at the beginning of the chapter.

* Recall that the contact triangle of ABC was defined in Chapter 2 as the triangle whose vertices are the contact points of the incircle with the sides of ABC.

Example 5.4. Find the circumradius and inradius of a triangle ABC with side lengths $AB = 13$, $BC = 14$, $CA = 15$.

Solution. First, we use Heron's formula to compute the area. Letting $a = 14$, $b = 15$, $c = 13$, we have $s = \frac{1}{2}(a + b + c) = 21$ and Heron's formula yields

$$\sqrt{s(s-a)(s-b)(s-c)} = \sqrt{21 \cdot 7 \cdot 6 \cdot 8} = 84.$$

Hence

$$[ABC] = \frac{abc}{4R} \Rightarrow R = \frac{abc}{4[ABC]} = \frac{13 \cdot 14 \cdot 15}{4 \cdot 84} = \frac{65}{8}.$$

Furthermore,

$$r = \frac{[ABC]}{s} = \frac{84}{21} = 4.$$ □

Of course, one would never see this type of computation on an olympiad, but this is just to illustrate a point. When doing computation, it is useful to be able to relate the quantities of a triangle to each other quickly. Areas provide a means to do this.

Problems for this Section

Problem 5.5. Show that $[ABC] = sr$. **Hint**: 462

Problem 5.6. In $\triangle ABC$ we have $AB = 13$, $BC = 14$, $CA = 15$. Find the length of the altitude from A onto $\overline{BC}$.

5.3 Trigonometry

We have already met the extended law of sines (Theorem 3.1), which states that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$

This is the first main trigonometric relation in a triangle. The second is the law of cosines, which we state below.

Theorem 5.7 (Law of Cosines). *Given a triangle ABC, we have*

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

Equivalently,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

Together the law of sines and the law of cosines form the backbone of trigonometric force. As we are about to see, these two in combination can single-handedly eradicate entire problems.

The way to do this is by thinking about **degrees of freedom**. Essentially, a statement in olympiad geometry has some number of parameters that can be selected, after which the rest of the diagram is uniquely determined, up to translation and rotation. For example, a triangle is determined by three parameters—for example, three sides, two sides and an included angle, or a side and two angles. Hence, we say that a generic triangle has three degrees of freedom.

For a subtler example, look at Problem 1.43 again.

Points A, B, C, D, E lie on a circle ω and point P lies outside the circle. The given points are such that (i) lines PB and PD are tangent to ω, (ii) P, A, C are collinear, and (iii) $\overline{DE} \parallel \overline{AC}$. Prove that $\overline{BE}$ bisects $\overline{AC}$.

How many degrees of freedom does this problem have? Suppose we drop the center O of the circle in the plane somewhere. We have one degree of freedom in picking its radius, and another degree of freedom in picking the distance OP. (Selecting the point P only gives one degree of freedom because we can rotate P about O arbitrarily without changing the figure.) At this point we can construct the tangents $\overline{PB}$ and $\overline{PD}$. We get one more degree of freedom in picking the point A on the circle, but then both C and E are determined. So in total, this problem has three degrees of freedom.

Why do we care? The point of trigonometry is to start with however many degrees of freedom are given, assign variables for each, and then blatantly pin down the remaining lengths and angles in terms of these variables. This is exactly what the law of cosines and the law of sines do.

Unfortunately, we also often obtain lots of ugly products of trigonometric expressions. This is where trigonometric identities come into play. Of course a reader is likely already familiar with the identities

$$\begin{aligned} 1 &= \sin^2\theta + \cos^2\theta \\ \sin(-\theta) &= -\sin\theta \\ \cos(-\theta) &= \cos\theta \\ \sin(\alpha+\beta) &= \sin\alpha\cos\beta + \sin\beta\cos\alpha \\ \cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta. \end{aligned}$$

The trickier identities are the so-called **product-to-sum** identities, which are indispensable for trigonometric calculation.

Proposition 5.8 (Product-Sum Identities). *For arbitrary α and β we have*

$$\begin{aligned} 2\cos\alpha\cos\beta &= \cos(\alpha-\beta) + \cos(\alpha+\beta) \\ 2\sin\alpha\sin\beta &= \cos(\alpha-\beta) - \cos(\alpha+\beta) \\ 2\sin\alpha\cos\beta &= \sin(\alpha-\beta) + \sin(\alpha+\beta). \end{aligned}$$

It is not necessary to memorize these because they are easy to rederive: just remember that the expansion of

$$\cos(x-y) \pm \cos(x+y)$$

has some cancellations. Changing the cosines to sines gives the other identities.

The product-sum identities let us repeatedly decompose messes obtained from a trigonometric siege into single sums. An example is the proof of Ptolemy's theorem, which follows this section.

5.4 Ptolemy's Theorem

There are some other non-trigonometric ways to relate side lengths when we have more than just a triangle. One often useful with cyclic quadrilaterals is Ptolemy's theorem[†].

Theorem 5.9 (Ptolemy's Theorem). *Let $ABCD$ be a cyclic quadrilateral. Then*

$$AB \cdot CD + BC \cdot DA = AC \cdot BD.$$

We are about to give a proof using trigonometry, but a more elegant proof appears in Chapter 8.

Before beginning our trigonometric attack, we should think about what to set as our variables. One might be tempted to set the lengths as variables, but this does little good. A second idea is to look at angles. Angles are nice because of the extended law of sines, which we can tie in to the circumradius. In fact, if we set $R = \frac{1}{2}$ as the radius of $(ABCD)$ (meaning we assume without loss of generality that we have diameter 1), we immediately get

$$AB = \sin \angle AXB$$

for any point X on the circumcircle. So it makes sense to use angles as variables.

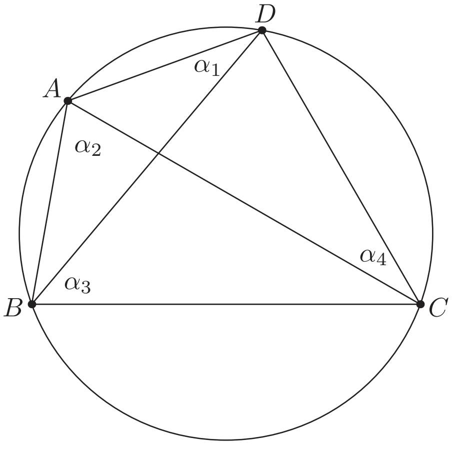

Figure 5.4A. A proof of Ptolemy's theorem.

A reasonable choice of our parameters is $\angle ADB$, $\angle BAC$, $\angle CBD$, $\angle DCA$. Most importantly, these four angles uniquely determine the diagram. This is really important, since otherwise we have no way of knowing if we have handled all the conditions. Note that there is actually a relation between these four angles; namely that they sum to 180°. We

[†] Ptolemy's theorem is actually an inequality: if A, B, C, D are four arbitrary points then $AB \cdot CD + BC \cdot DA \geq AC \cdot BD$, and equality holds if A, B, C, D lie on a circle or line in that order.

can use four variables anyways to preserve symmetry, but we need to keep this condition in mind as we proceed. Fortunately this particular condition is not so bad. If worst comes to worst, we can dump α_4 by replacing it with $180^\circ - (\alpha_1 + \alpha_2 + \alpha_3)$.

These are important remarks to make in general. Whenever you begin a calculation you need to think about degrees of freedom, and pick your variables to encompass all of them.

The other good part of this choice, of course, is that we get all the lengths we want from these angles immediately.

Proof. Let us denote α_1, α_2, α_3, and α_4 as the angles $\angle ADB$, $\angle BAC$, $\angle CBD$ and $\angle DCA$, and for convenience let us assume that the circumcircle of $ABCD$ has unit diameter. Then by the extended law of sines, we obtain

$$AB = \sin\alpha_1, \quad BC = \sin\alpha_2, \quad CD = \sin\alpha_3, \quad DA = \sin\alpha_4.$$

Furthermore,

$$AC = \sin\angle ABC = \sin(\alpha_3 + \alpha_4)$$

and

$$BD = \sin\angle DAB = \sin(\alpha_2 + \alpha_3)\,.$$

Note that we could have just as easily chosen $BD = \sin\angle BCD = \sin(\alpha_1 + \alpha_4)$. The quantities are equal, so the choice is irrelevant.

Now we just want to show that

$$\sin\alpha_1 \sin\alpha_3 + \sin\alpha_2 \sin\alpha_4 = \sin(\alpha_3 + \alpha_4)\sin(\alpha_2 + \alpha_3)$$

for $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 180^\circ$.

All the geometry is gone, so we appeal to Proposition 5.8 in order to deal with the products. We have that

$$\begin{aligned}
\sin\alpha_1 \sin\alpha_3 &= \frac{1}{2}(\cos(\alpha_1 - \alpha_3) - \cos(\alpha_1 + \alpha_3)) \\
\sin\alpha_2 \sin\alpha_4 &= \frac{1}{2}(\cos(\alpha_2 - \alpha_4) - \cos(\alpha_2 + \alpha_4)) \\
\sin(\alpha_2 + \alpha_3)\sin(\alpha_3 + \alpha_4) &= \frac{1}{2}(\cos(\alpha_2 - \alpha_4) - \cos(\alpha_2 + 2\alpha_3 + \alpha_4))\,.
\end{aligned}$$

We appear to be in pretty good shape here, because using our condition we find the cancellation

$$\cos(\alpha_1 + \alpha_3) + \cos(\alpha_2 + \alpha_4) = 0$$

on the left-hand side. We use the sum condition again to clean up the weird $\alpha_2 + 2\alpha_3 + \alpha_4$; we have

$$\cos(\alpha_2 + 2\alpha_3 + \alpha_4) = \cos(180^\circ - \alpha_1 + \alpha_3) = -\cos(\alpha_1 - \alpha_3)\,.$$

and now everything is clear. $\square$

It is important to notice the power of trigonometry here. Once all the geometry was gone, we knew we had something that had to be true; hence the problem reduced to making ends (expressions?) meet. Notice how the product-sum identities were used to deal with these resulting expressions.

It is deeply reassuring to know with full confidence that eventually the trigonometry will work out. The only downside is that sometimes the computations are too unwieldy to do by hand.

Actually, we can even refine Ptolemy's theorem as follows.

Theorem 5.10 (Strong Form of Ptolemy's Theorem). *In a cyclic quadrilateral $ABCD$ with $AB = a$, $BC = b$, $CD = c$, $DA = d$ we have*

$$AC^2 = \frac{(ac+bd)(ad+bc)}{ab+cd} \quad \text{and} \quad BD^2 = \frac{(ac+bd)(ab+cd)}{ad+bc}.$$

It is not hard to see that Ptolemy's theorem follows immediately from Theorem 5.10.

Let us briefly sketch two proofs. The first is to simply set

$$AC^2 = a^2 + b^2 - 2ab\cos\angle ABC = c^2 + d^2 - 2cd\cos\angle ADC$$

and then note that $\angle ADC + \angle ABC = 180^\circ$. With enough calculation this gives the result.

A second proof involves using the original Ptolemy's theorem on three cyclic quadrilaterals, where

(i) The first quadrilateral is $ABCD$, so its sides measure a, b, c, d in that order.
(ii) The second has sides measuring a, b, d, c in that order.
(iii) The third has sides measuring a, c, b, d in that order.

These all have the same circumradius, and one finds that there are only three distinct diagonal lengths. Applying the usual Ptolemy's theorem and doing some algebra then yields the conclusion. The details are left as an exercise.

A consequence of Ptolemy's theorem is the so-called Stewart's theorem, which we present here as a bit of trivia.

Theorem 5.11 (Stewart's Theorem). *Let ABC be a triangle. Let D be a point on $\overline{BC}$ and let $m = DB$, $n = DC$, $d = AD$. Then*

$$a(d^2 + mn) = b^2m + c^2n.$$

Often this is written in the form

$$man + dad = bmb + cnc$$

as a mnemonic—"a *man* and his *dad* put a *bomb* in the *sink*".

Proof. Let ray AD meet (ABC) again at P. By similar triangles we obtain

$$\frac{BP}{m} = \frac{b}{d} \text{ and } \frac{CP}{n} = \frac{c}{d}.$$

Furthermore, by power of a point we know that

$$DP = \frac{mn}{d}.$$

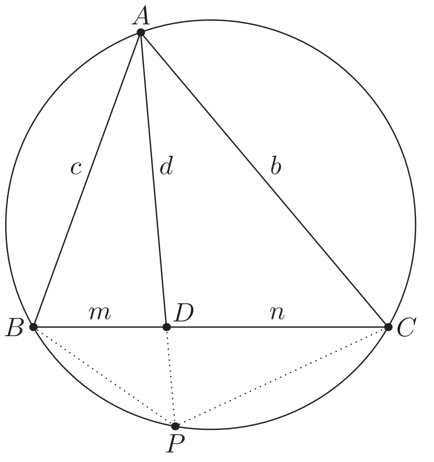

Figure 5.4B. Statement and proof of Stewart's theorem.

Now apply Ptolemy's theorem to obtain

$$BC \cdot AP = AC \cdot BP + AB \cdot CP$$

whence

$$a \cdot \left(d + \frac{mn}{d}\right) = b \cdot \frac{bm}{d} + c \cdot \frac{cn}{d}$$

which yields Stewart's theorem. □

Stewart's theorem can also be proved by using the law of cosines. One can check that

$$\frac{m^2 + d^2 - c^2}{2md} = \cos \angle ADB = -\cos \angle ADC = -\frac{n^2 + d^2 - b^2}{2nd}$$

and rearranging gives $m(n^2 + d^2 - b^2) + n(m^2 + d^2 - c^2) = 0$, or $a(mn + d^2) = b^2 m + c^2 n$.

Unlike Ptolemy's theorem, Stewart's theorem seldom sees use on olympiads. However, it features prominently on short-answer contests by providing a means to compute the length of a cevian.

Problem for this Section

Problem 5.12. Complete the synthetic proof above of Theorem 5.10, the stronger version of Ptolemy's theorem. **Hint**: 67

5.5 Example Problems

First we provide an example that illustrates the combination of Cartesian coordinates with length calculations. This problem was selected from the Harvard-MIT Math Tournament's Team Round in 2014.

Example 5.13 (Harvard-MIT Math Tournament 2014). Let ABC be an acute triangle with circumcenter O such that $AB = 4$, $AC = 5$, $BC = 6$. Let D be the foot of the altitude from A to $\overline{BC}$ and E be the intersection of lines AO and BC. Suppose that X is on $\overline{BC}$ between D and E such that there is a point Y on $\overline{AD}$ satisfying $\overline{XY} \parallel \overline{AO}$ and $\overline{YO} \perp \overline{AX}$. Determine the length of BX.

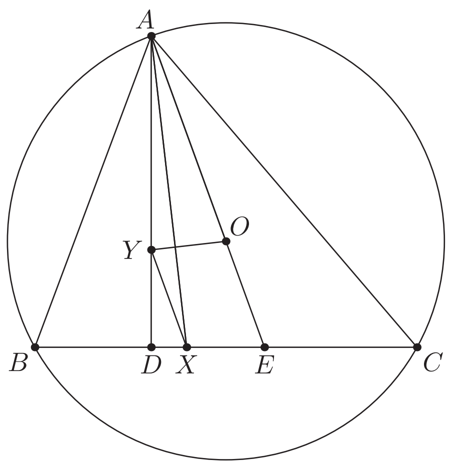

Figure 5.5A. Tossing on the coordinate plane with origin D.

This is a nice and difficult problem that could appear readily on the olympiad. Before we utterly spoil it, here is a quick sketch of the synthetic solution. Let ray AX meet (ABC) at P. First, show that the tangent to the circumcircle at A is concurrent with lines OY and BC. (This can be done with angle chasing.) Now use this to show that the tangent at P also passes through the concurrency point. This implies by Lemma 4.26 that $\overline{AX}$ is a symmedian; hence we obtain that

$$\frac{BX}{CX} = \left(\frac{AB}{AC}\right)^2$$

at which point we can easily compute BX.

Now let us exploit the fact that this problem is phrased computationally to provide a brute-force solution. Let us look at what conditions we have to decide how we might proceed.

- The point D is the foot of an altitude onto $\overline{BC}$.
- The point E is the intersection of a line through the circumcenter O and the side $\overline{BC}$.
- The points X and Y have a parallel condition and a perpendicularity condition.

Seeing right angles inspires us to use Cartesian coordinates. If so, where should we place the origin? The point D looks like a good candidate, as this lets us handle nicely the altitude, and makes the points A, B, C related to side lengths. In addition, the condition $\overline{XY} \parallel \overline{AO}$ is nicely encoded. (Actually one might notice that the point E does little in the problem. But it will be useful anyways for our computations.)

Solution to Example 5.13. First we need to compute AD. We can do this by using the area of ABC (obtained from Heron's formula); compute

$$AD = \frac{2[ABC]}{BC} = \frac{2}{6} \cdot \sqrt{\frac{15}{2} \cdot \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2}} = \frac{1}{3} \cdot \frac{15}{4}\sqrt{7} = \frac{5}{4}\sqrt{7}.$$

This makes $BD = \sqrt{4^2 - \frac{25}{16} \cdot 7} = \frac{9}{4}$ and subsequently $CD = 6 - \frac{9}{4} = \frac{15}{4}$. So we set

$$\begin{aligned} D &= (0, 0) \\ B &= (-9, 0) \\ C &= (15, 0) \\ A &= (0, 5\sqrt{7}). \end{aligned}$$

Here we are scaling the coordinate system up by a factor of four to ease computation (by eliminating fractions).

Next, we ought to compute O. We can compute the circumradius using

$$\frac{abc}{4R} = \frac{15}{4}\sqrt{7} \Rightarrow R = \frac{8}{\sqrt{7}}.$$

So the distance from O to BC is

$$\sqrt{\frac{8^2}{7} - 3^2} = \frac{1}{\sqrt{7}} = \frac{\sqrt{7}}{7}.$$

Also, noticing that O is directly overhead the midpoint of $\overline{BC}$, we can compute

$$O = \left(3, \frac{4}{7}\sqrt{7}\right)$$

in our coordinate system. (The extra factor of four again comes from our scaling.)

Next we need to compute E. We can do so using Theorem 4.22 (as $\overline{AD}$ and $\overline{AE}$ are isogonal), or by simply finding the x-intercept of the line AO. We do the latter. The slope of line AO is

$$\frac{5\sqrt{7} - \frac{4}{7}\sqrt{7}}{0 - 3} = -\frac{31}{21}\sqrt{7}$$

and hence the coordinate of E is

$$E = \left(\frac{5\sqrt{7}}{\frac{31}{21}\sqrt{7}}, 0\right) = \left(\frac{105}{31}, 0\right).$$

Now for a trick—we can encode the parallel condition by letting r denote the ratio between the lengths of $\overline{XY}$ and $\overline{AE}$. Therefore

$$X = \left(\frac{105}{31}r, 0\right) \text{ and } Y = \left(0, 5\sqrt{7} \cdot r\right).$$

(Similar triangles forever!) Now the condition $\overline{AX} \perp \overline{YO}$ is just a slope condition. We have

$$\begin{aligned}
-1 &= (\text{slope of } \overline{AX}) \cdot (\text{slope of } \overline{YO}) \\
&= \frac{5\sqrt{7} - 0}{0 - \frac{105}{31}r} \cdot \frac{\frac{4}{7}\sqrt{7} - 5\sqrt{7} \cdot r}{3 - 0} \\
&= \left(\frac{-31}{21r}\right)\left(\frac{4 - 35r}{3}\right) \\
\Rightarrow \frac{21r}{31} &= \frac{4 - 35r}{3} \\
\Rightarrow 63r &= 124 - 1085r \\
\Rightarrow r &= \frac{31}{287}.
\end{aligned}$$

We are home free—note that

$$X = \left(\frac{105}{31} \cdot \frac{31}{287}, 0\right) = \left(\frac{15}{41}, 0\right).$$

Hence, subtracting and scaling back gives

$$BX = \frac{1}{4}\left(\frac{15}{41} + 9\right) = \frac{96}{41}$$

and we are done. □

This is a typical coordinate solution. It is remarkable how little geometric insight was required after the first few lines—the rest was simply algebraic manipulations. In the context of olympiad problems, we generally have variables instead of the constants $a = 4$, $b = 5$, $c = 6$ that we did here.

Next, we provide an example of a trigonometric solution. This was problem four at the IMO 2009.

Example 5.14 (IMO 2009/4). Let ABC be a triangle with $AB = AC$. The angle bisectors of $\angle CAB$ and $\angle ABC$ meet the sides BC and CA at D and E, respectively. Let K be the incenter of triangle ADC. Suppose that $\angle BEK = 45^\circ$. Find all possible values of $\angle CAB$.

What makes this problem so ripe for calculation? Well, if we scale down the diagram (dropping a degree of freedom), then all points are determined by one angle... and then we have a constraint $\angle BEK = 45^\circ$. So up to scaling, this problem has zero degrees of freedom. This makes it pretty tempting to approach with computation.

First, we label all the angles in the figure. We choose to set $\angle DAC = 2x$, so that

$$\angle ACI = \angle ICD = 45^\circ - x.$$

Here I is the incenter of ABC. In that case $\angle AIE = \angle DIC$ (why?), but $\angle DIC = \frac{1}{2}\angle BIC = x + 45^\circ$, hence $\angle AIE = x + 45^\circ$. Some more angle chasing gives $\angle KEC = 3x$.

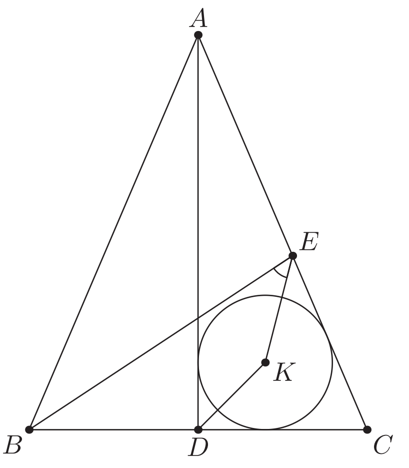

Figure 5.5B. Example 5.14.

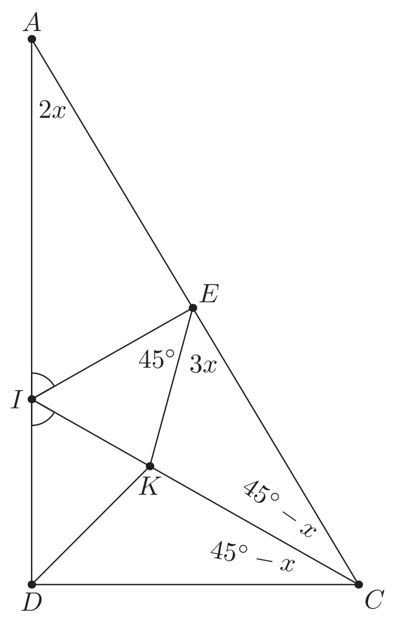

Figure 5.5C. Setup for a trigonometric computation.

Having chased all the angles we want, we need a relationship. We can find it by considering the side ratio $\frac{IK}{KC}$. Using the angle bisector theorem, we can express this in terms of triangle IDC; however we can also express it in terms of triangle IEC. This gives us an algebraic equation to solve.

Solution to Example 5.14. Let I be the incenter, and set $\angle DAC = 2x$ (so that $0^\circ < x < 45^\circ$). From $\angle AIE = \angle DIC$, it is easy to compute

$$\angle KIE = 90^\circ - 2x, \ \angle ECI = 45^\circ - x, \ \angle IEK = 45^\circ, \ \angle KEC = 3x.$$

Hence by the law of sines, we can obtain

$$\frac{IK}{KC} = \frac{\sin 45^\circ \cdot \frac{EK}{\sin(90^\circ - 2x)}}{\sin(3x) \cdot \frac{EK}{\sin(45^\circ - x)}} = \frac{\sin 45^\circ \sin(45^\circ - x)}{\sin(3x)\sin(90^\circ - 2x)}.$$

Also, by the angle bisector theorem on $\triangle IDC$, we have

$$\frac{IK}{KC} = \frac{ID}{DC} = \frac{\sin(45^\circ - x)}{\sin(45^\circ + x)}.$$

Equating these and cancelling $\sin(45^\circ - x) \neq 0$ gives

$$\sin 45^\circ \sin(45^\circ + x) = \sin 3x \sin(90^\circ - 2x).$$

Applying the product-sum formula (again, we are just trying to break down things as much as possible), this just becomes

$$\cos(x) - \cos(90^\circ + x) = \cos(5x - 90^\circ) - \cos(90^\circ + x)$$

or $\cos x = \cos(5x - 90^\circ)$.

At this point we are basically done; the rest is making sure we do not miss any solutions and write up the completion nicely. One nice way to do this is by using product-sum in reverse as

$$0 = \cos(5x - 90^\circ) - \cos x = 2\sin(3x - 45^\circ)\sin(2x - 45^\circ).$$

This way we merely consider the two cases

$$\sin(3x - 45^\circ) = 0 \text{ and } \sin(2x - 45^\circ) = 0.$$

Notice that $\sin\theta = 0$ if and only θ is an integer multiple of 180°. Using the bound $0^\circ < x < 45^\circ$, it is easy to see that that the permissible values of x are $x = 15^\circ$ and $x = \frac{45}{2}^\circ$. As $\angle A = 4x$, this corresponds to $\angle A = 60^\circ$ and $\angle A = 90^\circ$, the final answer. □

Our last quick example is a problem from the 2004 Chinese Girl's Math Olympiad.

Example 5.15 (CGMO 2004/6). Let ABC be an acute triangle with O as its circumcenter. Line AO intersects BC at D. Points E and F are on $\overline{AB}$ and $\overline{AC}$ respectively such that A, E, D, F are concyclic. Prove that the length of the projection of line segment EF on side $\overline{BC}$ does not depend on the positions of E and F.

In our figure we have denoted the projections of E and F by X and Y, respectively.

How might we approach this problem computationally? Our goal is to get everything in terms of the quantities in a triangle, and we have one degree of freedom in our problem.

We are interested in the length XY, so it seems natural to write

$$XY = BC - (BX + CY)$$

because the lengths BX and CY seem easy to calculate—they are the legs of a right triangle. Actually, we may even just write

$$BX = BE\cos B \quad \text{and} \quad CY = CF\cos C.$$

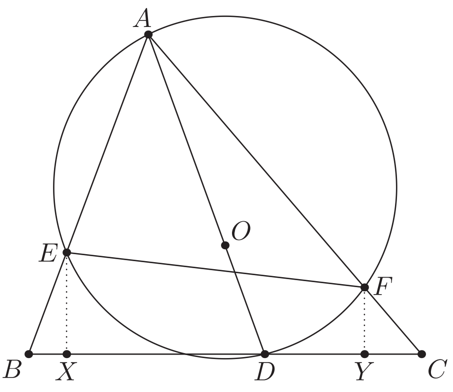

Figure 5.5D. Show that the length of $\overline{XY}$ depends only on ABC.

We do not have to worry about $\cos B$ anymore, and so we can go for BE. Naturally, we reach to power of a point, as we have

$$BE \cdot BA = BT \cdot BD$$

where we have defined T as the second intersection of our cyclic quadrilateral with side $\overline{BC}$ (this is a sort of proxy point). Similarly, $CF \cdot CA = CD \cdot CT$. Now we have a natural choice for encoding our degree of freedom: define $u = BT$, $v = CT$ with $u + v = a$. Then we can compute the lengths BD and CD by whatever means we choose, directly evaluate $BX + CY$, and hope we get something constant.

Solution to Example 5.15. Recall that $\angle BAD = \angle BAO = 90^\circ - C$ and $\angle CAD = \angle CAO = 90^\circ - B$. First, we can compute using the law of sines that

$$\frac{BD}{CD} = \frac{\sin \angle BAD \cdot \frac{AB}{\sin \angle ADB}}{\sin \angle CAD \cdot \frac{AC}{\sin \angle ADC}} = \frac{c \cos C}{b \cos B}.$$

Now let X and Y denote the feet of E and F onto $\overline{BC}$ and T the second intersection of (AEF) with $\overline{BC}$. Let $u = BT$, $v = CT$ where $u + v = a$; we have

$$\begin{aligned} BX + CY &= BE \cos B + CF \cos C \\ &= \frac{u \cdot BD}{c} \cos B + \frac{v \cdot CD}{b} \cos C \\ &= \cos B \cos C \left(\frac{BD}{c \cos C} u + \frac{CD}{b \cos B} v \right). \end{aligned}$$

Because

$$\frac{BD}{c \cos C} = \frac{CD}{b \cos B}$$

and

$$u + v = a$$

we see that $BX + CY$ does not depend on the choice of u and v, completing the solution. □

5.6 Problems

Another good source of practice problems are any problems in the previous sections that you failed to solve synthetically, since you should have some insight into the problem's structure. See how you can use computation to make up for missed synthetic observations. (This advice applies to the next two chapters as well.)

Problem 5.16 (Star Theorem)**.** Let $A_1A_2A_3A_4A_5$ be a convex pentagon. Suppose rays A_2A_3 and A_5A_4 meet at the point X_1. Define X_2, X_3, X_4, X_5 similarly. Prove that

$$\prod_{i=1}^{5} X_iA_{i+2} = \prod_{i=1}^{5} X_iA_{i+3}$$

where the indices are taken modulo 5. (See Figure 5.6A.) **Hints**: 407 448 **Sol:** p.251

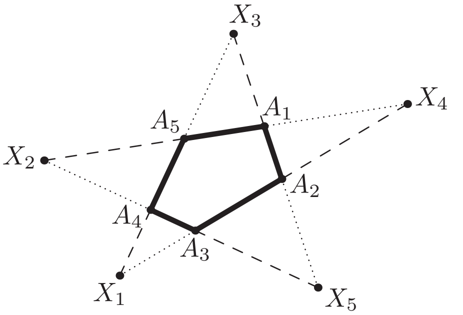

Figure 5.6A. Star theorem—the product of the dashed segments is the product of the dotted ones.

Problem 5.17. Let ABC be a triangle with inradius r. If the exradii [‡] of ABC are r_A, r_B, r_C, show that the triangle has area $\sqrt{r \cdot r_A \cdot r_B \cdot r_C}$. **Hint**: 38

Problem 5.18 (APMO 2013/1). Let ABC be an acute triangle with altitudes $\overline{AD}$, $\overline{BE}$ and $\overline{CF}$, and let O be the center of its circumcircle. Show that the segments OA, OF, OB, OD, OC, OE dissect the triangle ABC into three pairs of triangles that have equal areas. **Hints**: 162 678

Problem 5.19 (EGMO 2013/1). The side BC of the triangle ABC is extended beyond C to D so that $CD = BC$. The side CA is extended beyond A to E so that $AE = 2CA$. Prove that if $AD = BE$ then triangle ABC is right-angled. **Hints**: 202 275

Problem 5.20 (Harvard-MIT Math Tournament 2013). Let triangle ABC satisfy $2BC = AB + AC$ and have incenter I and circumcircle ω. Let D be the intersection of AI and ω (with A, D distinct). Prove that I is the midpoint of $\overline{AD}$. **Hints**: 372 477

Problem 5.21 (USAMO 2010/4). Let ABC be a triangle with $\angle A = 90°$. Points D and E lie on sides AC and AB, respectively, such that $\angle ABD = \angle DBC$ and $\angle ACE = \angle ECB$.

[‡] Recall from Chapter 2 that the A-exradius of $\triangle ABC$ is the radius of the excircle opposite A. The B and C exradii are defined similarly.

Segments BD and CE meet at I. Determine whether or not it is possible for segments AB, AC, BI, ID, CI, IE to all have integer lengths. **Hints**: 437 603 565 **Sol:** p.252

Problem 5.22 (Iran Olympiad 1999). Let I be the incenter of triangle ABC and let ray AI meet the circumcircle of ABC at D. Denote the feet of the perpendiculars from I to lines BD and CD by E and F, respectively. If $IE + IF = \frac{1}{2}AD$, calculate $\angle BAC$. **Hints**: 359 610 365 479 **Sol:** p.252

Problem 5.23 (CGMO 2002/4). Circles Γ_1 and Γ_2 interest at two points B and C, and $\overline{BC}$ is the diameter of Γ_1. Construct a tangent line to circle Γ_1 at C intersecting Γ_2 at another point A. Line AB meets Γ_1 again at E and line CE meets Γ_2 again at F. Let H be an arbitrary point on segment AF. Line HE meets Γ_2 again at G, and $\overline{BG}$ meets $\overline{AC}$ at D.

Prove that

$$\frac{AH}{HF} = \frac{AC}{CD}.$$

Hints: 452 62 344 219

Problem 5.24 (IMO 2007/4). In triangle ABC the bisector of angle BCA intersects the circumcircle again at R, the perpendicular bisector of $\overline{BC}$ at P, and the perpendicular bisector of $\overline{AC}$ at Q. The midpoint of $\overline{BC}$ is K and the midpoint of $\overline{AC}$ is L. Prove that the triangles RPK and RQL have the same area. **Hints**: 457 291 139 161

Problem 5.25 (JMO 2013/5). Quadrilateral $XABY$ is inscribed in the semicircle ω with diameter $\overline{XY}$. Segments AY and BX meet at P. Point Z is the foot of the perpendicular from P to line XY. Point C lies on ω such that line XC is perpendicular to line AZ. Let Q be the intersection of segments AY and XC. Prove that

$$\frac{BY}{XP} + \frac{CY}{XQ} = \frac{AY}{AX}.$$

Hints: 622 476 299 656

Problem 5.26 (CGMO 2007/5). Point D lies inside triangle ABC such that $\angle DAC = \angle DCA = 30^\circ$ and $\angle DBA = 60^\circ$. Point E is the midpoint of segment $\overline{BC}$. Point F lies on segment $\overline{AC}$ with $AF = 2FC$. Prove that $\overline{DE} \perp \overline{EF}$. **Hints**: 483 690 180 542 693

Problem 5.27 (ISL 2011/G1). Let ABC be an acute triangle. Let ω be a circle whose center L lies on the side BC. Suppose that ω is tangent to $\overline{AB}$ at B' and $\overline{AC}$ at C'. Suppose also that the circumcenter O of triangle ABC lies on the shorter arc $B'C'$ of ω. Prove that the circumcircle of ABC and ω meet at two points. **Hints**: 13 87 93 500 60 **Sol:** p.253

Problem 5.28 (IMO 2001/1). Consider an acute-angled triangle ABC. Let P be the foot of the altitude of triangle ABC issuing from the vertex A, and let O be the circumcenter of triangle ABC. Assume that $\angle C \geq \angle B + 30^\circ$. Prove that $\angle A + \angle COP < 90^\circ$. **Hints**: 619 246 522

Problem 5.29 (IMO 2001/5). Let ABC be a triangle. Let $\overline{AP}$ bisect $\angle BAC$ and let $\overline{BQ}$ bisect $\angle ABC$, with P on $\overline{BC}$ and Q on $\overline{AC}$. If $AB + BP = AQ + QB$ and $\angle BAC = 60^\circ$, what are the angles of the triangle? **Hints**: 43 71 441 226 **Sol:** p.254

Problem 5.30 (IMO 2001/6). Let $a > b > c > d$ be positive integers and suppose that

$$ac + bd = (b + d + a - c)(b + d - a + c).$$

Prove that $ab + cd$ is not prime.[§] **Hints**: 166 555 523 429 515 **Sol:** p.255

[§] IMO 2001 was a *strange* year.

CHAPTER 6

Complex Numbers

> As long as algebra and geometry have been separated, their progress have been slow and their uses limited; but when these two sciences have been united, they have lent each mutual forces, and have marched together towards perfection.
>
> Joseph Louis Lagrange

In this chapter, we demonstrate the use of complex numbers to solve problems in geometry. We develop some background in the first three sections. The real geometry starts in Section 6.4, when the unit circle appears.

6.1 What is a Complex Number?

Recall some facts from high school algebra. A **complex number** is a number of the form

$$z = a + bi$$

where a and b are real numbers and $i^2 = -1$. The real number a is called the **real part**, denoted $\operatorname{Re}(z)$. The set of all complex numbers is denoted $\mathbb{C}$.

We also know that every complex number can be expressed in **polar form** as

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

where r is a nonnegative real number and θ is a real number. (The formula $e^{i\theta} = \cos\theta + i\sin\theta$ is a famous result known as *Euler's formula*.) A diagram may make this clearer; much like in the xy-plane, every complex number can be plotted in the **complex plane** at a point (a, b). See Figure 6.1A.

The **magnitude** of $z = a + bi = re^{i\theta}$, denoted $|z|$, is equal to r, or equivalently,

$$|z| = \sqrt{a^2 + b^2}.$$

The number θ is called the **argument** of z, denoted $\arg z$. It is the angle measured counterclockwise from the real axis, as shown in Figure 6.1A. Except in the special case $z = 0$, the fact that r is a positive real implies θ is unique up to shifting by 360°. (As a specific example, $\cos 50^\circ + i\sin 50^\circ = \cos 410^\circ + i\sin 410^\circ$.) Therefore, for the rest of this chapter we take these arguments modulo 360°.

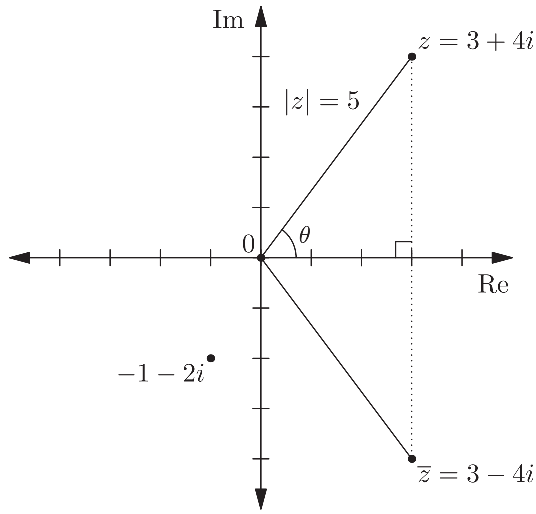

Figure 6.1A. The numbers $z = 3 + 4i$ and $-1 - 2i$ are plotted in the complex plane; $\overline{z} = 3 - 4i$ is the conjugate of z.

Finally, the **complex conjugate** of z (or just **conjugate**) is the number

$$\overline{z} = a - bi = re^{-i\theta}.$$

Pictorially, it represents the reflection of z over the real axis.

The conjugate has many nice properties: it behaves well with respect to basically every operation. For example, whenever w and z are complex numbers, we have

$$\overline{w + z} = \overline{w} + \overline{z}, \quad \overline{w - z} = \overline{w} - \overline{z}, \quad \overline{w \cdot z} = \overline{w} \cdot \overline{z}, \quad \overline{w/z} = \overline{w}/\overline{z},$$

and so on. (Verify these.) This lets us write, for instance,

$$\overline{\left(\frac{z - a}{b - a}\right)} = \frac{\overline{z} - \overline{a}}{\overline{b} - \overline{a}}$$

and similarly reduce other arbitrarily complicated expressions. Another important relation is that for any complex number z,

$$|z|^2 = z\overline{z}.$$

This is easy to prove and, as we see later, extremely useful.

Throughout this chapter, we let A denote the point in the complex plane that corresponds to a complex number a, and adopt similar conventions for the other letters, with lowercase letters denoting complex numbers, and uppercase letters denoting points.

6.2 Adding and Multiplying Complex Numbers

Complex numbers can be viewed a lot like vectors (u, v). We simply think about them in the component form $u + vi$ and note that adding them corresponds to vector addition.

This means that all the additive structure of vectors (see Appendix A.3) carries over. For example,

1. The midpoint M of $\overline{AB}$ is $m = \frac{1}{2}(a+b)$.
2. Three points A, B, C are collinear if (and only if) $c = \lambda a + (1-\lambda)b$ for some real number λ.
3. The centroid G of a triangle ABC is $g = \frac{1}{3}(a+b+c)$.
4. A quadrilateral $ABCD$ is a parallelogram if and only if $a+c = b+d$.

And so on. In particular, adding a complex number corresponds to translation, just as in vectors.

However, complex numbers have some additional structure—they can be *multiplied*. The multiplication is particularly powerful. The key is that if $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, then $z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$, which implies

$$|z_1 z_2| = |z_1|\,|z_2| \text{ and } \arg z_1 z_2 = \arg z_1 + \arg z_2 \text{ for all } z_1, z_2 \in \mathbb{C}.$$

We remind the reader that here (and throughout this chapter) we are taking $\arg z$ modulo 360°. So the above equality really means $\arg z_1 z_2 \equiv \arg z_1 + \arg z_2 \pmod{360^\circ}$.

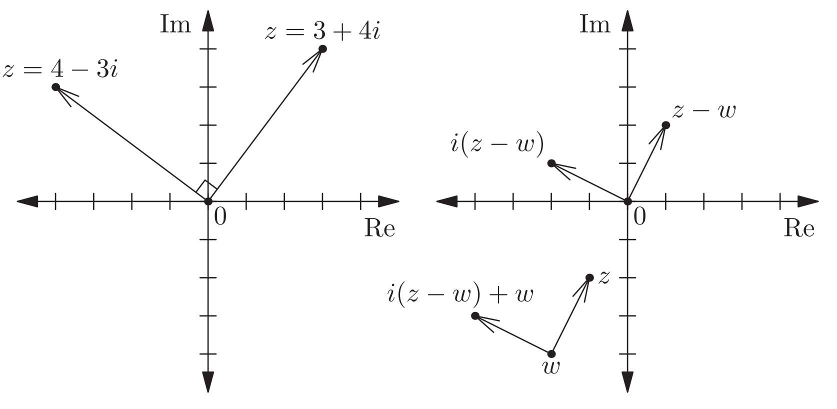

Figure 6.2A. Rotating by 90° is just multiplying by i.

Example 6.1. Multiplying by i is equivalent to rotating by 90° counterclockwise around the origin.

Proof. Just notice that $|i| = 1$ and $\arg i = \frac{1}{2}\pi = 90^\circ$. □

This is fine and well, but how do we rotate around arbitrary points? Suppose we want to rotate $z = -1-2i$ by 90° counterclockwise about the point $w = -2-4i$. The answer is simple; we translate the entire diagram so that $w \mapsto 0$ (by subtracting w). We then multiply by i, and then translate back. In equations, this looks like

$$z \mapsto i(z-w)+w.$$

Pictorially, this is much more intuitive. See Figure 6.2A.

We can generalize further to any complex number other than i. For any complex number w and nonzero α, the map

$$z \mapsto \alpha(z - w) + w$$

is a **spiral similarity**. That means it is a map that rotates by $\arg \alpha$ and dilates by $|\alpha|$; it is a composition of a rotation and a homothety. Spiral similarity is discussed in more detail in Section 10.1.

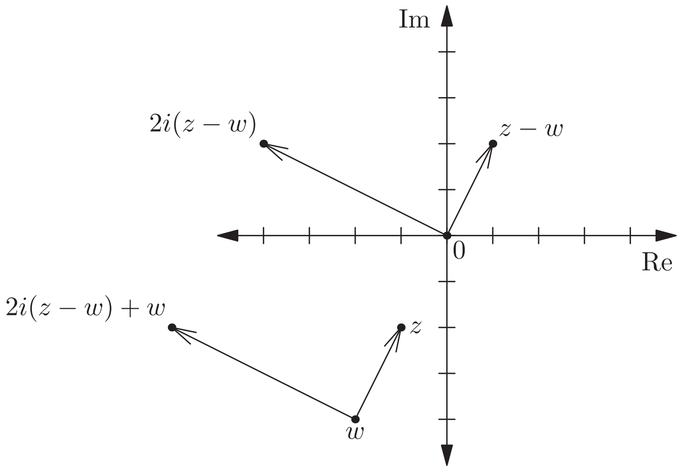

Figure 6.2B. A spiral similarity $z \mapsto 2i(z - w) + w$. It rotates by 90° and dilates by a factor of 2.

We can do even more, as the following lemma shows.

Lemma 6.2 (Complex Reflection). *Let W be the reflection of Z over a given $\overline{AB}$. Then*

$$w = \frac{(a - b)\overline{z} + \overline{a}b - a\overline{b}}{\overline{a} - \overline{b}}.$$

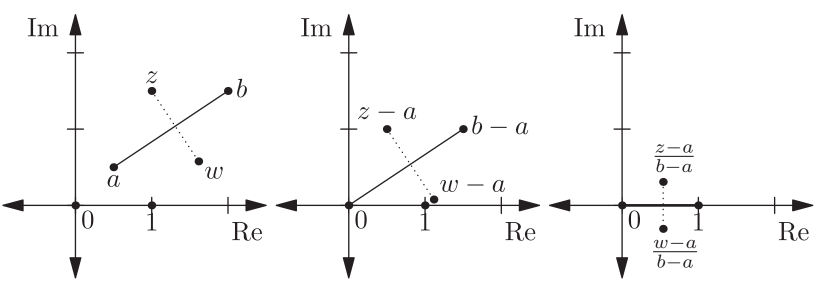

Figure 6.2C. Reflecting about $\overline{AB}$.

Proof. We remarked earlier that the map $z \mapsto \overline{z}$ was a reflection across the real axis. We would like to do something similar with a and b.

Figure 6.2C essentially gives away the proof. We first shift the entire diagram by subtracting a. Then, we apply a spiral similarity through dividing by the shifted $b - a$, so that the line we are trying to reflect across becomes the real axis. Under these two transformations

$$z \mapsto \frac{z-a}{b-a} \text{ and } w \mapsto \frac{w-a}{b-a}.$$

But these two are now conjugate! That is,

$$\frac{z-a}{b-a} = \overline{\left(\frac{w-a}{b-a}\right)}.$$

This is better expressed as

$$\frac{w-a}{b-a} = \overline{\left(\frac{z-a}{b-a}\right)} = \frac{\overline{z}-\overline{a}}{\overline{b}-\overline{a}}.$$

Solving for w and doing some computation we obtain

$$w = \frac{a(\overline{b}-\overline{a}) + (b-a)(\overline{z}-\overline{a})}{\overline{b}-\overline{a}} = \frac{(a-b)\overline{z} + \overline{a}b - a\overline{b}}{\overline{a}-\overline{b}}$$

as desired. □

Problem for this Section

Lemma 6.3. *Show that the foot of the altitude from Z to $\overline{AB}$ is given by*

$$\frac{(\overline{a}-\overline{b})z + (a-b)\overline{z} + \overline{a}b - a\overline{b}}{2(\overline{a}-\overline{b})}.$$

6.3 Collinearity and Perpendicularity

Let us first state two obvious facts about the complex conjugate.

Proposition 6.4 (Properties of Complex Conjugates). *Let z be a complex number.*

(a) $z = \overline{z}$ if and only if z is a real number.
*(b) $z + \overline{z} = 0$ if and only if z is **pure imaginary**; that is, $z = ri$ for some real number r.*

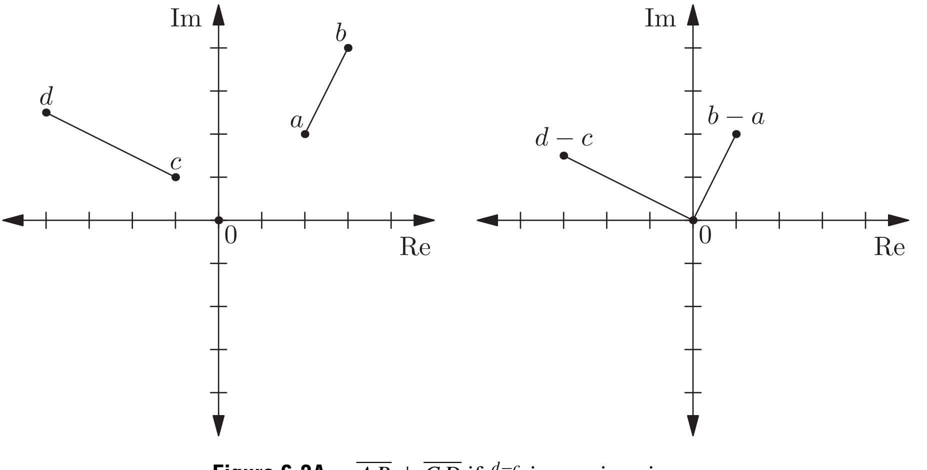

Figure 6.3A. $\overline{AB} \perp \overline{CD}$ if $\frac{d-c}{b-a}$ is pure imaginary.

First, let us develop a criterion for when $\overline{AB} \perp \overline{CD}$. Consider four complex numbers a, b, c, d and look at the corresponding vectors $b - a$ and $d - c$.

Since $\arg z/w = \arg z - \arg w$, we observe that the $d - c$ and $b - a$ are perpendicular precisely when their arguments differ by $\pm 90^\circ$; that is, when $\frac{d-c}{b-a}$ is pure imaginary. In terms of conjugates, we deduce the following.

Lemma 6.5 (Perpendicularity Criterion). *The complex numbers a, b, c, d have the property $\overline{AB} \perp \overline{CD}$ if and only if*

$$\frac{d-c}{b-a} + \overline{\left(\frac{d-c}{b-a}\right)} = 0.$$

By effectively the same means, we can arrive at a collinearity criterion.

Lemma 6.6. *Complex numbers z, a, b are collinear if and only if*

$$\frac{z-a}{z-b} = \overline{\left(\frac{z-a}{z-b}\right)}.$$

The proof is essentially the same as that of Lemma 6.5; we consider the displacements $z - a$ and $z - b$, and hope that their quotient is a real number. The details are left as an exercise.

However, you might notice that that Lemma 6.6 is not symmetric, which seems disappointing. Actually, we ran into the exact same issue in Section 5.1, when we were trying to find a nice criterion for collinear points. Surprisingly, the same method works here as well.

Theorem 6.7 (Complex Shoelace Formula). *If a, b, c are complex numbers, then the signed area of triangle ABC is given by*

$$\frac{i}{4}\begin{vmatrix} a & \overline{a} & 1 \\ b & \overline{b} & 1 \\ c & \overline{c} & 1 \end{vmatrix}.$$

In particular, the points a, b, c are collinear if and only if the determinant is zero.

Here the signed area is the convention described in Section 5.1. This formula actually follows from the standard shoelace formula; write $a = a_x + a_y i$, $b = b_x + b_y i$, and $c = c_x + c_y i$, and apply the shoelace formula to a, b, c. The details, which consist entirely of linear algebra, are left as an exercise.

Problem for this Section

Problem 6.8. Prove Lemma 6.6.

6.4 The Unit Circle

Up until now we have had conjugates in many of our expressions. We now show how to handle them, closing the gap between olympiad geometry and complex numbers.

In the complex plane, the **unit circle** is the set of complex numbers z with $|z| = 1$; that is, it is a circle centered at 0 with radius 1. We have the following.

Proposition 6.9. *For any z on the unit circle, $\overline{z} = \frac{1}{z}$.*

This follows from $z\overline{z} = |z|^2$, where we take advantage of the fact that $|z| = 1$. That means we can now compute conjugates in terms of the original complex numbers. Here are two examples of straightforward applications.

Example 6.10. If a, b, c, and x lie on the unit circle, then $ax + bc = 0$ if and only if $\overline{AX} \perp \overline{BC}$.

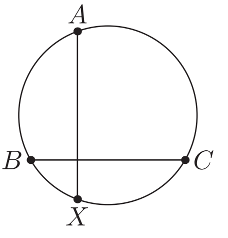

Figure 6.4A. $AX \perp BC$ implies $ax + bc = 0$.

Proof. By Lemma 6.3 we know that $\overline{AX} \perp \overline{BC}$ is equivalent to

$$0 = \frac{x-a}{b-c} + \overline{\left(\frac{x-a}{b-c}\right)} = \frac{x-a}{b-c} + \frac{\overline{x}-\overline{a}}{\overline{b}-\overline{c}}.$$

Applying $\overline{a} = \frac{1}{a}$, this is equal to

$$\begin{aligned} &\frac{x-a}{b-c} + \frac{\frac{1}{x}-\frac{1}{a}}{\frac{1}{b}-\frac{1}{c}} \\ &= \frac{x-a}{b-c} + \frac{\frac{a-x}{xa}}{\frac{c-b}{bc}} \\ &= \frac{x-a}{b-c}\left(1 + \frac{xa}{bc}\right). \end{aligned}$$

Since a, b, c, x are distinct, the first quantity is nonzero; hence we obtain $\frac{xa}{bc} = -1$, equivalent to $ax + bc = 0$. □

We now present a refinement of Lemma 6.3. It is used extremely frequently, so remember it!

Lemma 6.11 (Complex Foot). *If a and b, $a \neq b$, are on the unit circle and z is an arbitrary complex number, then the foot from Z to AB is given by*

$$\frac{1}{2}(a + b + z - ab\overline{z}).$$

Proof. Putting $\overline{a} = \frac{1}{a}$ and $\overline{b} = \frac{1}{b}$ in Lemma 6.3 we get

$$\frac{1}{2}\left(z + \frac{(a-b)\overline{z} + \frac{b}{a} - \frac{a}{b}}{\frac{1}{a} - \frac{1}{b}}\right) = \frac{1}{2}(z + a + b - ab\overline{z}).$$ □

In the limiting case $a = b$, we obtain the foot from z to the tangent at a.

We are now in a position to derive some useful results, independent of any geometry we know. The following beautiful result is critical, and really shows how powerful complex numbers are.

Lemma 6.12 (Complex Euler Line). *Let ABC be a triangle, and assume a, b, c lie on the unit circle. Then*

(a) The circumcenter is $o = 0$.
(b) The centroid is $g = \frac{1}{3}(a + b + c)$.
(c) The orthocenter is $h = a + b + c$.

In particular, the points O, G, H are collinear in a $1 : 2$ ratio.

Proof. The fact that $o = 0$ is obvious, since we set the circumcircle of ABC as the unit circle. The fact that $g = \frac{1}{3}(a + b + c)$ follows by interpreting the complex numbers as vectors.

Let h be the orthocenter. There are many ways to prove that $h = a + b + c$, and we present the solution which uses no geometry. Because $\overline{AH} \perp \overline{BC}$ we know by Lemma 6.5 that

$$\begin{aligned} 0 &= \frac{h - a}{b - c} + \frac{\overline{h} - \overline{a}}{\overline{b} - \overline{c}} \\ &= \frac{h - a}{b - c} + \frac{\overline{h} - \frac{1}{a}}{\frac{1}{b} - \frac{1}{c}} \\ &= \frac{h - a}{b - c} - bc\frac{\overline{h} - \frac{1}{a}}{b - c}. \end{aligned}$$

Therefore,

$$\begin{aligned} bc\left(\overline{h} - \frac{1}{a}\right) &= h - a \\ \Rightarrow abc\overline{h} - bc &= ah - a^2 \\ \Rightarrow abc\overline{h} - ah &= bc - a^2. \end{aligned}$$

We can derive similar equations from $\overline{BH} \perp \overline{CA}$ and $\overline{CH} \perp \overline{AB}$. Hence, we wish to solve the system of equations

$$\begin{aligned} abc\overline{h} - ah &= bc - a^2 \\ abc\overline{h} - bh &= ca - b^2 \\ abc\overline{h} - ch &= ab - c^2. \end{aligned}$$

Just subtract the first two equations to get

$$(b - a)h = b^2 - a^2 + bc - ca = (b - a)(a + b + c).$$

Since $b \neq a$, we obtain $h = a + b + c$. It is not too hard to verify that this is indeed a solution to all three equations, and so we have established that the orthocenter exists and has coordinates $h = a + b + c$. Finally, since $h = 3g$ it follows that O, G, H are collinear with $OH = 3OG$; this establishes the Euler line. $\square$

Example 6.13 (Nine-Point Circle). If a, b, c lie on the unit circle, and H is the orthocenter of $\triangle ABC$, the point $n_9 = \frac{1}{2}(a+b+c)$ is a distance of $\frac{1}{2}$ from the midpoint of $\overline{BC}$, the midpoint of $\overline{AH}$, and the foot from A to $\overline{BC}$.

Proof. First, we check the distance to the midpoint of $\overline{BC}$. It is

$$\left| n_9 - \frac{b+c}{2} \right| = \left| \frac{a}{2} \right| = \frac{1}{2}\,|a| = \frac{1}{2}.$$

Then we check the distance to the midpoint of $\overline{AH}$. It is

$$\left| n_9 - \frac{1}{2}(a + (a+b+c)) \right| = \left| -\frac{a}{2} \right| = \frac{1}{2}.$$

Finally, we check the distance to the foot of the altitude is also $\frac{1}{2}$. By Lemma 6.11, this is the point $\frac{1}{2}\left(a+b+c-\frac{bc}{a}\right)$. So

$$\left| n_9 - \frac{1}{2}\left(a+b+c-\frac{bc}{a}\right) \right| = \left| \frac{1}{2}\frac{bc}{a} \right| = \frac{1}{2}\frac{|b|\,|c|}{|a|} = \frac{1}{2}.$$

That was easy. $\square$

We hope this convinces you that setting (ABC) as the unit circle is an extremely potent technique. After all, it just trivialized a large portion of Chapter 3.

Problem for this Section

Problem 6.14. (Lemma 1.17) Let H be the orthocenter of $\triangle ABC$. Let X be the reflection of H over $\overline{BC}$ and Y the reflection over the midpoint of $\overline{BC}$. Prove that X and Y lie on (ABC), and $\overline{AY}$ is a diameter.

6.5 Useful Formulas

Here are some other useful formulas. First we provide a criterion for when four points are concyclic.

Theorem 6.15 (Concyclic Complex Numbers). *Let a, b, c, d be distinct complex numbers, not all collinear. Then A, B, C, D are concyclic if and only if*

$$\frac{b-a}{c-a} \div \frac{b-d}{c-d}$$

is a real number.

The proof is left as an exercise. (Actually, we see in Chapter 9 that if A, B, C, D are indeed cyclic, then this is the cross ratio of the four points $ABCD$.)

In the same spirit as the complex shoelace formula (Theorem 6.7) is the following similarity criterion. To show $\triangle ABC$ and $\triangle XYZ$ are similar with the same orientation, most people attempt to prove $\frac{c-a}{b-a} = \frac{z-x}{y-x}$ or some similar variant. Actually, a symmetric version of this formula* exists.

Theorem 6.16 (Complex Similarity). *Two triangles ABC and XYZ are directly similar if and only if*

$$0 = \begin{vmatrix} a & x & 1 \\ b & y & 1 \\ c & z & 1 \end{vmatrix}.$$

Proof. The triangles are similar if and only if

$$\frac{c-a}{b-a} = \frac{z-x}{y-x}.$$

One can check this is equivalent to the determinant being equal to zero. □

Now, here is the complete form for the intersection of two lines.

Theorem 6.17 (Complex Intersection). *If lines AB and CD are not parallel then their intersection is given by*

$$\frac{(\overline{a}b - a\overline{b})(c-d) - (a-b)(\overline{c}d - c\overline{d})}{(\overline{a} - \overline{b})(c-d) - (a-b)(\overline{c} - \overline{d})}.$$

In particular, if $|a| = |b| = |c| = |d| = 1$ then this simplifies to

$$\frac{ab(c+d) - cd(a+b)}{ab - cd}.$$

Proof. Solve the system of equations

$$0 = \begin{vmatrix} z & \overline{z} & 1 \\ a & \overline{a} & 1 \\ b & \overline{b} & 1 \end{vmatrix} = \begin{vmatrix} z & \overline{z} & 1 \\ c & \overline{c} & 1 \\ d & \overline{d} & 1 \end{vmatrix}.$$

This is not much fun, but you get the result with enough patience. If $\overline{a} = \frac{1}{a}$ and its analogous forms are substituted, then we get the second expression. □

It is worth noting that the conjugate of the second expression in Theorem 6.17 is $\frac{a+b-c-d}{ab-cd}$.

This theorem exemplifies why the choice of unit circle is extremely important—the formula becomes far simpler when a, b, c, d are on the unit circle. In general, the more points that lie on the unit circle, the better, because the conjugates become simple reciprocals rather than complicated expressions.

Nonetheless, the fully general intersection formula is sometimes useful as well. In particular, if $d = 0$ the expression is actually somewhat tamer. It is also often possible to apply translations before applying the theorem to simplify the computation; see Example 6.26 for an instance of this.

You can even get the intersection of two circles—sort of. Here is the statement, just for fun. We give the proof in Section 10.1, but you are welcome to prove it now.

* What happens below when we take $x = \overline{a}$, $y = \overline{b}$, $z = \overline{c}$?

Lemma 6.18. *Suppose X and Y are the intersection points of two circles. Points A and B lie on the first circle, C and D on the second, such that lines AC and BD pass through X. Then*

$$y = \frac{ad - bc}{a + d - b - c}.$$

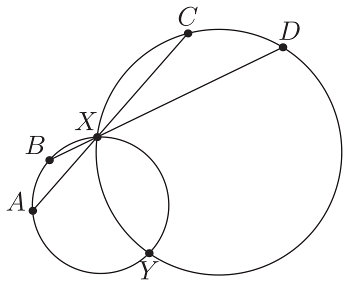

Figure 6.5A. Handling circle intersections in the complex plane.

Finally, one common configuration which complex numbers handles well is the intersection of two tangents to the unit circle.

Lemma 6.19 (Complex Tangent Intersection). *Let A and B be points on the unit circle with $a + b \neq 0$. Then*

$$\frac{2ab}{a+b} = \frac{2}{\overline{a} + \overline{b}}$$

is the intersection point of the tangents at A and B.

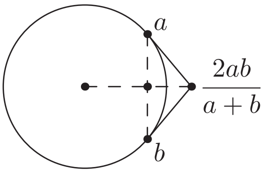

Figure 6.5B. Intersecting two tangents in the complex plane.

Proof. Consult Figure 6.5B. Let M be the midpoint of $\overline{AB}$ and P be the desired intersection point. It is not hard to show that $OM \cdot OP = 1$ (where $o = 0$) by similar triangles. Hence $|m|\,|p| = 1$.

We claim this implies $\overline{m} \cdot p = 1$. Indeed, the magnitudes are correct, and because O, M, P are collinear, the argument is zero as well. Hence

$$p = \frac{1}{\overline{m}} = \frac{2}{\overline{a} + \overline{b}} = \frac{2}{\frac{1}{a} + \frac{1}{b}} = \frac{2ab}{a+b}. \qquad \square$$

Problems for this Section

Problem 6.20. Prove Theorem 6.16. **Hint**: 217

Problem 6.21. Prove that the complex shoelace formula (Theorem 6.7) follows from Theorem 5.1. **Hint**: 644

Problem 6.22. Let ABC be a triangle with orthocenter H and let P be a point on (ABC).

(a) Show that the Simson line (Lemma 1.48) exists, i.e., that the feet from P onto $\overline{AB}$, $\overline{BC}$, $\overline{CA}$ are collinear.
(b) Establish Lemma 4.4; that is, show that the Simson line at P bisects $\overline{PH}$.

Hint: 535

6.6 Complex Incenter and Circumcenter

Two other complex setups worth mentioning are the incenter and the circumcenter.

Let us start with a different question. If b and c lie on the unit circle, what is the midpoint of minor arc $\widehat{BC}$? It might be tempting to say $\sqrt{bc}$, but unfortunately taking a square root of a complex number raises problems. For example, consider

$$(1-i)^2 = (i-1)^2 = -2i.$$

We can no longer take a "positive root" because there is no notion of "positive" or "negative" complex numbers.

Fortunately there is a way around this. If we set $b = w^2$ and $c = v^2$, then we can designate one of vw or $-vw$ as the midpoint of arc BC. This motivates the following lemma.

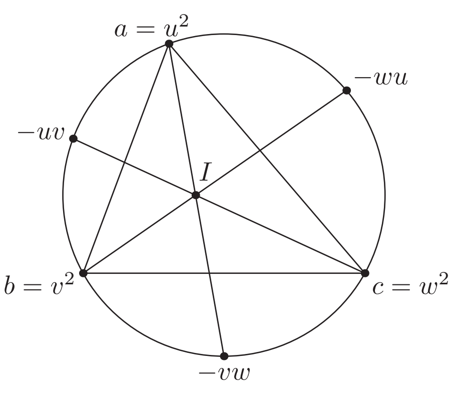

Figure 6.6A. Lemma 1.42.

Lemma 6.23 (Complex Incenter). *Given ABC on the unit circle, it is possible to pick complex numbers u, v, w such that*

(a) $a = u^2, b = v^2, c = w^2$, and
(b) the midpoint of arc $\widehat{BC}$ not containing A is $-vw$; the analogous midpoints opposite B and C are $-wu$ and $-uv$.

In this case the incenter I is given by $-(uv + vw + wu)$.

Proof. Proving the first two claims involves cumbersome algebra; you can probably skip it but we include it for completeness. By rotating the triangle, we may assume that $a = 1$. Now set $u = -1$, and let v and w represent the desired midpoints. We claim this is the desired (u, v, w). See Figure 6.6B.

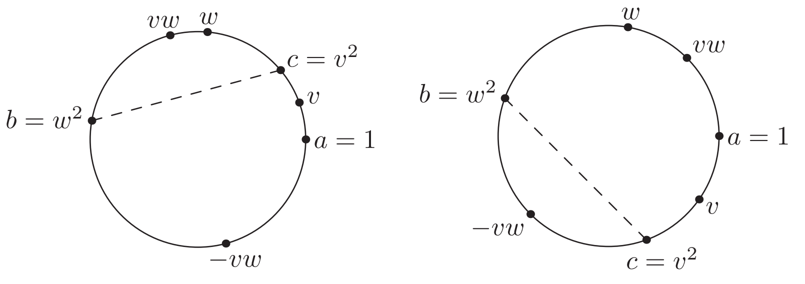

Figure 6.6B. Proving the midpoints of arcs formula.

By construction, $b = w^2$ and $c = v^2$. It remains to show that $-vw$ actually lies on the arc $\widehat{BC}$ not containing A (as opposed to the midpoint of the arc containing A). This is equivalent to showing vw and $a = 1$ lie on the same side of $\overline{BC}$.

Now for some boring details. We consider two cases, which can be extended to cover all situations.

- Both v and w have arguments between 0 and π. Let β be the argument of v, and γ the argument of w. Assume without loss of generality $\beta > \gamma$. Then $\arg a = 0$, $\arg c = 2\gamma$, $\arg vw = \beta + \gamma$ and $\arg w^2 = 2\beta$, where

$$0 < 2\gamma < \beta + \gamma < 2\beta < 2\pi.$$

  This establishes the conclusion.
- w has argument β and v has argument $-\gamma$, where $0 < \beta, \gamma < \pi$. Let $\theta = \beta - \gamma$ be the argument of vw and without loss of generality assume $\theta > 0$. We also have $\arg a = 0$, $\arg w^2 = \min\{2\beta, 2\pi - 2\gamma\}$, and $\arg v^2 = \max\{2\beta, 2\pi - 2\gamma\}$, where

$$0 < \theta < \min\{2\beta, 2\pi - 2\gamma\} < \max\{2\beta, 2\pi - 2\gamma\} < 2\pi$$

  as needed.

For the more interesting part, recall Lemma 1.42. We see I is the orthocenter of the triangle with vertices $-vw$, $-wu$, $-uv$, and hence is $-(uv + vw + wu)$ since all three vertices lie on the unit circle. □

Note also that $|u| = |v| = |w| = 1$, so in particular $\overline{u} = \frac{1}{u}$, $\overline{v} = \frac{1}{v}$, $\overline{w} = \frac{1}{w}$ still hold.

The last formula we present is the formula for the circumcenter. While we usually set the circumcenter we care about to zero, it is actually possible to compute the circumcenter of an arbitrary triangle, although it is not always feasible to do this computation.

Lemma 6.24 (Complex Circumcenter). *The circumcenter of a triangle XYZ is given by the quotient*

$$\begin{vmatrix} x & x\overline{x} & 1 \\ y & y\overline{y} & 1 \\ z & z\overline{z} & 1 \end{vmatrix} \div \begin{vmatrix} x & \overline{x} & 1 \\ y & \overline{y} & 1 \\ z & \overline{z} & 1 \end{vmatrix}.$$

In particular, if $z = 0$ then the above expression equals

$$\frac{xy\left(\overline{x} - \overline{y}\right)}{\overline{x}y - x\overline{y}}.$$

Proof. Let P be the circumcenter of $\triangle XYZ$ and R the circumradius. We have

$$R^2 = |x - p|^2 = (x - p)(\overline{x} - \overline{p})$$

implying

$$\overline{x}p + x\overline{p} + R^2 = p\overline{p} + x\overline{x}.$$

Hence, we obtain the system of equations

$$\begin{aligned} x\overline{p} + \overline{x}p + R^2 - p\overline{p} &= x\overline{x} \\ y\overline{p} + \overline{y}p + R^2 - p\overline{p} &= y\overline{y} \\ z\overline{p} + \overline{z}p + R^2 - p\overline{p} &= z\overline{z}. \end{aligned}$$

By Cramer's Rule (Theorem A.4), we can view $\overline{p}$, p, and $R^2 - p\overline{p}$ as the unknowns (surprise!) to get

$$p = \begin{vmatrix} x & x\overline{x} & 1 \\ y & y\overline{y} & 1 \\ z & z\overline{z} & 1 \end{vmatrix} \div \begin{vmatrix} x & \overline{x} & 1 \\ y & \overline{y} & 1 \\ z & \overline{z} & 1 \end{vmatrix}$$

as required. $\square$

It is often useful to shift the points x, y, z to clear out common terms before applying the circumcenter formula. In particular, one can shift z to zero before evaluating the determinant, which simplifies the computation significantly (but breaks the symmetry). In this case the circumcenter is given by

$$z + \frac{-x'y'(\overline{x}' - \overline{y}')}{x'\overline{y}' - \overline{x}'y'}$$

where $x' = x - z$ and $y' = y - z$.

6.7 Example Problems

First, a classical result on the nine-point circle.

Proposition 6.25 (The Feuerbach Tangency). *The incircle and the nine-point circle of a (non-equilateral) triangle are tangent to each other. (The point of tangency is called the* ***Feuerbach point****.)*

Suppose we wish to prove this using complex numbers. Firstly, how do we handle the tangent condition? Circles are not particularly nice in complex numbers, so perhaps our best bet is to try lengths. If I and N_9 are the incenter and nine-point center, then it would suffice to prove

$$IN_9 = \frac{1}{2}R - r \quad \text{or equivalently} \quad 2IN_9 = R - 2r$$

since the nine-point circle has radius $\frac{1}{2}R$.

Actually, does the right-hand side look familiar? According to Lemma 2.22, we have $R - 2r = \frac{1}{R}IO^2$, where O is the circumcenter. That means we simply want to prove that

$$R \cdot 2IN_9 = IO^2.$$

Now we are in business. If we toss this on the complex plane with $R = 1$, all we have to do is compute some absolute values.

Seeing the incenter, let us put $A = x^2$, $B = y^2$, $C = z^2$ as in Lemma 6.23. Note in particular that $R = 1$. Then the incenter is given by $-(xy + yz + zx)$ while the nine-point center is given by $\frac{1}{2}\left(x^2 + y^2 + z^2\right)$. Evidently we get that

$$2IN_9 = 2\left|\frac{1}{2}\left(x^2 + y^2 + z^2\right) - [-(xy + yz + zx)]\right| = |x + y + z|^2.$$

A miracle occurs—we manage to get a perfect square! Now we just compute IO^2, and of course we should get exactly the same thing and we can call it a day. We find

$$IO^2 = |-(xy + yz + zx) - 0|^2 = |xy + yz + zx|^2.$$

Oh wait, those are not actually the same.

The problem has now reduced to showing that $|x + y + z|^2 = |xy + yz + zx|^2$, which might seem unexpected. Fortunately, squares of absolute values reduce to just conjugates. The left hand side is merely

$$(x + y + z)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right)$$

while the right hand side is

$$(xy + yz + zx)\left(\frac{1}{xy} + \frac{1}{yz} + \frac{1}{zx}\right).$$

These are both equal to $\frac{(x+y+z)(xy+yz+zx)}{xyz}$, so we are done.

Solution to Proposition 6.25. Using Lemma 6.23 we put complex numbers x^2, y^2, z^2, and $-(xy + yz + zx)$ for A, B, C, I respectively. Let N_9 be the center of the nine-point circle and let O be the circumcenter. Notice that

$$\begin{aligned} 2IN_9 &= 2\left|\frac{1}{2}(x^2+y^2+z^2) - [-(xy+yz+zx)]\right| \\ &= |x+y+z|^2 \\ &= |xy+yz+zx|^2 \\ &= IO^2 \\ &= R(R-2r) \\ &= R-2r \end{aligned}$$

where R and r are the circumradius and inradius, respectively. (We have $R = 1$ because we are on the unit circle.) It follows that $IN_9 = \frac{1}{2}R - r$ and hence that the circles are tangent. □

For our second example, we examine a problem from a USA team selection test. We present two solutions, one entirely computational (requiring basically no geometric skill at all) and one that only minimally touches on complex numbers.

Example 6.26 (USA TSTST 2013/1). Let ABC be a triangle and D, E, F be the midpoints of arcs BC, CA, AB on the circumcircle. Line ℓ_a passes through the feet of the perpendiculars from A to $\overline{DB}$ and $\overline{DC}$. Line m_a passes through the feet of the perpendiculars from D to $\overline{AB}$ and $\overline{AC}$. Let A_1 denote the intersection of lines ℓ_a and m_a. Define points B_1 and C_1 similarly. Prove that triangles DEF and $A_1B_1C_1$ are similar to each other.

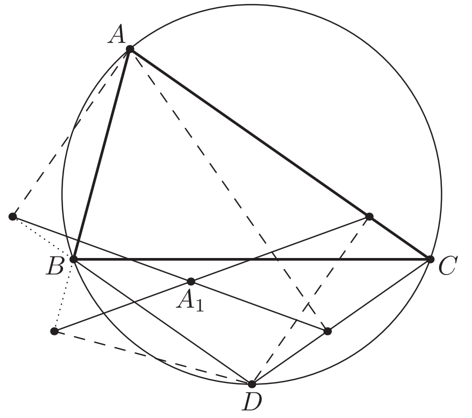

Figure 6.7A. The first problem of the 2013 TSTST.

What makes this problem good for complex numbers? First, there are loads of points all on a single circle, (ABC), and we will almost certainly choose that as the unit circle. The perpendiculars are also great here, because we are dropping altitudes to the chords of the circle, so we can use Lemma 6.11. Thirdly, there is a lot of symmetry—after we compute A_1 it is straightforward to compute B_1 and C_1. And finally, the similarity is a condition we know how to deal with.

Down to business. We want to compute A_1. In our usual notation, we see that the foot from D to $\overline{AB}$ (which we denote by P_1) is given by

$$p_1 = \frac{1}{2}\left(a + b + d - ab\overline{d}\right).$$

If we set $a = x^2$ and so on, along with $d = -yz$, then this reduces to

$$p_1 = \frac{1}{2}\left(x^2 + y^2 - yz + \frac{x^2 y}{z}\right).$$

Similarly, the foot from D to $\overline{AC}$ is

$$p_2 = \frac{1}{2}\left(x^2 + z^2 - yz + \frac{x^2 z}{y}\right).$$

We now consider the other half of the story. The feet from A to $\overline{BD}$ and $\overline{CD}$, which we call Q_1 and Q_2, are none other than

$$q_1 = \frac{1}{2}\left(x^2 + y^2 - yz + \frac{y^3 z}{x^2}\right) \text{ and } q_2 = \frac{1}{2}\left(x^2 + z^2 - yz + \frac{yz^3}{x^2}\right).$$

Now we need to construct A_1. Unfortunately, trying to apply Theorem 6.17 directly looks painful (but feasible). We can do better by noticing that there are a lot of repeated terms in these four points. So here is the idea: consider the map

$$\tau : \alpha \mapsto 2\alpha - (x^2 + y^2 + z^2 - yz).$$

Where did that come from? The key observation is that τ preserves intersections, since it just combines a dilation and a translation. That means that if A_1 is the intersection of lines P_1P_2 and Q_1Q_2, then $\tau(A_1)$ represents the intersection of lines $\tau(P_1)\tau(P_2)$ and $\tau(Q_1)\tau(Q_2)$. And now it is pretty clear why we chose that map. Everything simplifies beautifully under τ. We got rid of the $\frac{1}{2}$s and trimmed out all the extra fat with the $x^2 - yz$ terms that were appearing everywhere. Thus,

$$\tau(p_1) = -z^2 + \frac{x^2 y}{z} \qquad \tau(p_2) = -y^2 + \frac{x^2 z}{y}$$
$$\tau(q_1) = -z^2 + \frac{y^3 z}{x^2} \qquad \tau(q_2) = -y^2 + \frac{z^3 y}{x^2}.$$

This looks much friendlier—still messy, maybe, but we can make it through. Abbreviating x' for $\tau(x)$, and applying Theorem 6.17, we see that $\tau(a_1)$ equals

$$\frac{\left(p_1'\overline{p}_2' - \overline{p}_1'p_2'\right)\left(q_1' - q_2'\right) - \left(q_1'\overline{q}_2' - \overline{q}_1'q_2'\right)\left(p_1' - p_2'\right)}{\left(\overline{p}_1' - \overline{p}_2'\right)\left(q_1' - q_2'\right) - \left(p_1' - p_2'\right)\left(\overline{q}_1' - \overline{q}_2'\right)}.$$

At this point you might want to estimate how long this computation is going to take—it is starting to look pretty lengthy. Fortunately the time limit for this test was 4.5 hours for three problems. This looks like it might be a 15 or 20 minute computation, which is really not a bad investment at all.

We take this calculation one bit at a time. First,

$$p_1'\overline{p}_2' - \overline{p}_1'p_2' = \left(-z^2 + \frac{x^2y}{z}\right)\left(-\frac{1}{y^2} + \frac{y}{x^2z}\right) - \left(y^2 + \frac{x^2z}{y}\right)\left(-\frac{1}{z^2} + \frac{z}{x^2y}\right).$$

A couple of remarks. Notice you can save some effort by noticing that $\tau(p_1)\tau(\overline{p}_2)$ and $\tau(p_2)\tau(\overline{p}_1)$ just switch y and z. That way we only need to expand once. Also, notice how all terms have the same degree. When your expression has this property, you can use degrees as a quick way to catch obvious errors.

Now, expanding gives

$$\begin{aligned}p_1'\overline{p}_2' - \overline{p}_1'p_2' &= \left(\frac{z^2}{y^2} + \frac{y^2}{z^2} - \frac{x^2}{yz} - \frac{yz}{x^2}\right) \\ &\quad - \left(\frac{y^2}{z^2} + \frac{y^2}{z^2} - \frac{x^2}{yz} - \frac{yz}{x^2}\right) \\ &= 0.\end{aligned}$$

It looks like we will not need $\tau(q_1) - \tau(q_2)$ after all. We then evaluate

$$\begin{aligned}q_1'\overline{q}_2' - \overline{q}_1'q_2' &= \left(-z^2 + \frac{y^3z}{x^2}\right)\left(-\frac{1}{y^2} + \frac{x^2}{yz^3}\right) \\ &\quad - \left(-y^2 + \frac{yz^3}{x^2}\right)\left(-\frac{1}{z^2} + \frac{x^2}{y^3z}\right) \\ &= \left(\frac{z^2}{y^2} - \frac{yz}{x^2} - \frac{x^2}{yz} + \frac{y^2}{z^2}\right) \\ &\quad - \left(\frac{y^2}{z^2} - \frac{yz}{x^2} - \frac{x^2}{yz} + \frac{y^2}{z^2}\right) \\ &= 0.\end{aligned}$$

So $\tau(a_1) = 0$, a big surprise. (Usually it does not turn out this well.) For just a dozen lines of algebra we obtain

$$\tau(a_1) = 0 \Rightarrow a_1 = \frac{1}{2}\left(x^2 + y^2 + z^2 - yz\right).$$

Do we need to do the same for B_1 and C_1? Of course not. We simply exploit symmetry to get

$$b_1 = \frac{1}{2}\left(x^2 + y^2 + z^2 - zx\right).$$

$$c_1 = \frac{1}{2}\left(x^2 + y^2 + z^2 - xy\right).$$

Now we just need to show that this is similar to triangle DEF, which has vertices $-yz$, $-zx$, $-xy$. One can do this quite painlessly by appealing to Theorem 6.16. However, one can simply note that A_1, B_1, C_1 are the midpoints of the segments joining $x^2+y^2+z^2$ to each of D, E, F. This solves the problem.

We promised a mostly synthetic solution, though. An observant reader has probably by now noticed that $x^2+y^2+z^2 = a+b+c$ is the orthocenter of ABC. Hence A_1 is the midpoint of $\overline{DH}$. Does this configuration look familiar now?

Solution to Example 6.26. Let H be the orthocenter of ABC.

Firstly, m_a is the Simson line from D onto ABC, so it passes through the midpoint M_1 of $\overline{DH}$ by Lemma 4.4. Now let H_A be the orthocenter of $\triangle DBC$. Since ℓ_a is the Simson line of A onto BCD, it passes through the midpoint of $\overline{DH_A}$, say M_2.

We claim that these midpoints are the same. Indeed, in the language of complex numbers,

$$m_1 = \frac{(a+b+c)+d}{2} = \frac{a+(b+c+d)}{2} = m_2.$$

Hence A_1 is the midpoint of $\overline{DH}$. Similarly, B_1 is the midpoint of $\overline{EH}$ and C_1 is the midpoint of $\overline{FH}$. It follows that H is the center of a homothety taking $A_1B_1C_1$ onto DEF, completing the problem. □

Notice that we never actually used the fact that D was a midpoint of arc AB in the above solution. In fact, it is totally irrelevant. The problem holds true for any D, E, F on the circumcircle.

The point $\frac{1}{2}(a+b+c+d)$ for a cyclic quadrilateral $ABCD$ is called the **Euler point** or the **anticenter** of the cyclic quadrilateral. Note that as a corollary of the above calculations, we find that the Simson lines from A onto $\triangle BCD$, B onto $\triangle CDA$, C onto $\triangle DAB$ and D onto $\triangle ABC$ all pass through the anticenter.

For our third example, we select a problem from the USAMO 2012. This one is more straightforward, especially with our knowledge of the determinant.

Example 6.27 (USAMO 2012/5). Let P be a point in the plane of $\triangle ABC$, and γ a line passing through P. Let A', B', C' be the points where the reflections of lines PA, PB, PC with respect to γ intersect lines BC, AC, AB respectively. Prove that A', B', C' are collinear.

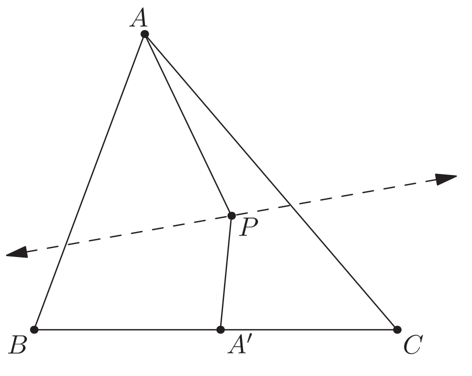

Figure 6.7B. USAMO 2012—reflecting lines over sides.

We might be tempted to set (ABC) as the unit circle again, but that would make the reflections through an arbitrary P quite gory. A better idea is to use the reflections to our advantage rather than avoid them—let us set γ as the real axis, so that the reflection of A across γ has coordinate $\overline{a}$. Of course, we may as well set $p = 0$ at this point.

With this setup, the rest is a computation. Note that determinants heavily simplify our calculation.

Solution to Example 6.27. Let P be the origin of the complex plane (meaning $p = 0$) and γ be the real axis. Now notice that A' is the intersection of lines bc and $p\overline{a}$. Applying the formula for the intersection of lines gives

$$a' = \frac{a(\overline{b}c - b\overline{c})}{(\overline{b} - \overline{c})a - (b - c)\overline{a}}.$$

Also,

$$\overline{a}' = \frac{\overline{a}(b\overline{c} - \overline{b}c)}{(b - c)\overline{a} - (\overline{b} - \overline{c})a}.$$

Considering the cyclic quantities, the area of $a'b'c'$ is a multiple of

$$\begin{vmatrix} \frac{a(\overline{b}c-b\overline{c})}{(\overline{b}-\overline{c})a-(b-c)\overline{a}} & \frac{\overline{a}(b\overline{c}-\overline{b}c)}{(b-c)\overline{a}-(\overline{b}-\overline{c})a} & 1 \\ \frac{b(\overline{c}a-c\overline{a})}{(\overline{c}-\overline{a})b-(c-a)\overline{b}} & \frac{\overline{b}(c\overline{a}-\overline{c}a)}{(c-a)\overline{b}-(\overline{c}-\overline{a})b} & 1 \\ \frac{c(\overline{a}b-a\overline{b})}{(\overline{a}-\overline{b})c-(a-b)\overline{c}} & \frac{\overline{c}(a\overline{b}-\overline{a}b)}{(a-b)\overline{c}-(\overline{a}-\overline{b})c} & 1 \end{vmatrix}.$$

This is actually a multiple of

$$\begin{vmatrix} a & \overline{a} & \frac{(\overline{b}-\overline{c})a-(b-c)\overline{a}}{\overline{b}c-b\overline{c}} \\ b & \overline{b} & \frac{(\overline{c}-\overline{a})b-(c-a)\overline{b}}{\overline{c}a-c\overline{a}} \\ c & \overline{c} & \frac{(\overline{a}-\overline{b})c-(a-b)\overline{c}}{\overline{a}b-a\overline{b}} \end{vmatrix}.$$

But now if we evaluate by minors, the denominators $\overline{b}c - b\overline{c}$ exactly cancel out with the resulting determinants, and we get

$$\sum_{\text{cyc}} \frac{(\overline{b} - \overline{c})a - (b - c)\overline{a}}{\overline{b}c - b\overline{c}} \cdot \begin{vmatrix} b & \overline{b} \\ c & \overline{c} \end{vmatrix} = \sum_{\text{cyc}} \left(a\overline{b} - a\overline{c} + c\overline{a} - b\overline{a}\right) = 0$$

as desired. (Here, the "cyclic sum" is as defined in Section 0.3.) □

We finish with a cute lemma about equilateral triangles in the complex plane.

Lemma 6.28 (Complex Equilateral Triangles). *Let ABC be a triangle. It is equilateral if and only if $a^2 + b^2 + c^2 = ab + bc + ca$.*

Proof. Let $u = a - b$, $v = b - c$, $w = c - a$. Notice that ABC is equilateral if and only if u, v, w are the roots of some cubic $z^3 - \alpha = 0$. (Why?) So we actually consider the polynomial

$$(z - u)(z - v)(z - w).$$

Expanding and noting $u + v + w = 0$, we have that it is

$$z^3 + (uv + vw + wu)z - uvw.$$

Hence ABC is equilateral if and only if $uv + vw + wu = 0$.

The rest is algebra. Rewrite the given as

$$a^2 + b^2 + c^2 = ab + bc + ca,$$

or equivalently,

$$0 = (a - b)^2 + (b - c)^2 + (c - a)^2 = u^2 + v^2 + w^2.$$

Standard manipulation with symmetric sums now gives us

$$0 = (u + v + w)^2 = u^2 + v^2 + w^2 + 2(uv + vw + wu).$$

So $uv + vw + wu = 0$ if and only if $a^2 + b^2 + c^2 = ab + bc + ca$, as desired. □

6.8 When (Not) to use Complex Numbers

In this section we echo some of the comments made above in the examples.

First, let us mention briefly what types of problems are NOT good candidates for complex numbers. The worst enemy of complex numbers is multiple circles. Complex numbers give control over the unit circle, but offer little help with handling any other circles. Intersections of arbitrary lines are also unwieldy (to say nothing of arbitrary circumcenters or incenters).

However, if most of the points can be coaxed into lying on a single circle, then we are in good shape. Moreover, if a central triangle features prominently on this circle, we have already seen that we can deal with its triangle centers. Indeed one of the most common techniques is to set (ABC) as the unit circle. This has the added bonus of exploiting any symmetry in the problem.

Finally, you should always look for synthetic observations to simplify a complex numbers solution. One attitude I like to use when solving a geometry problem is to use synthetic techniques until a problem is either solved or reduced to something that is readily susceptible to computation.

6.9 Problems

Problem 6.29. Give a proof of the inscribed angle theorem using complex numbers. **Hints**: 506 343

Lemma 6.30 (Complex Chord). *Show that a point P lies on a chord $\overline{AB}$ of the unit circle if and only if $p + ab\overline{p} = a + b$.* **Hint**: 86 **Sol:** p.256

Problem 6.31. Let $ABCD$ be a cyclic quadrilateral. Let H_A, H_B, H_C, H_D denote the orthocenters of triangles BCD, CDA, DAB, and ABC, respectively. Prove that $\overline{AH_A}$, $\overline{BH_B}$, $\overline{CH_C}$, and $\overline{DH_D}$ concur. **Hint**: 132

Problem 6.32. Let $ABCD$ be a quadrilateral circumscribed around a circle with center I. Prove that I lies on the line joining the midpoints of $\overline{AC}$ and $\overline{BD}$. **Hints**: 526 395 **Sol:** p.257

Problem 6.33 (Chinese TST 2011). Let ABC be a triangle, and let A', B', C' be points on its circumcircle, diametrically opposite A, B, C, respectively. Let P be any point inside ABC and let D, E, F be the feet of the altitudes from P onto $\overline{BC}$, $\overline{CA}$, $\overline{AB}$, respectively. Let X, Y, Z denote the reflections of A', B', C' over D, E, F, respectively.

Show that triangles XYZ and ABC are similar to each other. **Hints**: 141 149

Proposition 6.34 (Napoleon's Theorem). *Let ABC be a triangle and erect equilateral triangles on sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ outside of ABC with centers O_A, O_B, O_C. Prove that $\triangle O_A O_B O_C$ is equilateral and that its center coincides with the centroid of triangle ABC.* **Hints**: 380 237 558

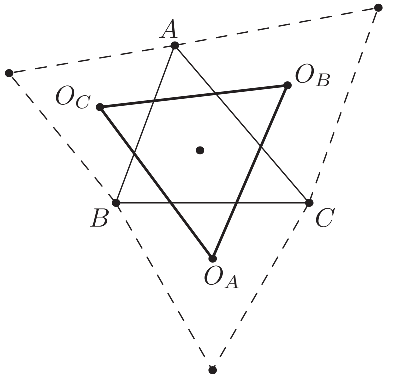

Figure 6.9A. Napoleon's theorem.

Problem 6.35 (USAMO 2015/2). Quadrilateral $APBQ$ is inscribed in circle ω with $\angle P = \angle Q = 90^\circ$ and $AP = AQ < BP$. Let X be a variable point on segment $\overline{PQ}$. Line AX meets ω again at S (other than A). Point T lies on arc AQB of ω such that $\overline{XT}$ is perpendicular to $\overline{AX}$. Let M denote the midpoint of chord $\overline{ST}$. As X varies on segment $\overline{PQ}$, show that M moves along a circle. **Hints**: 133 361 316 283 **Sol:** p.258

Problem 6.36 (MOP 2006). Point H is the orthocenter of triangle ABC. Points D, E, and F lie on the circumcircle of triangle ABC such that $\overline{AD} \parallel \overline{BE} \parallel \overline{CF}$. Points S, T, and U are the respective reflections of D, E, and F across the lines BC, CA, and AB. Prove that S, T, U, and H are concyclic. **Hints**: 313 173 513 **Sol:** p.259

Problem 6.37 (USA January TST for IMO 2014). Let $ABCD$ be a cyclic quadrilateral, and let E, F, G, and H be the midpoints of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$, respectively. Let W, X, Y, and Z be the orthocenters of triangles AHE, BEF, CFG, and DGH, respectively. Prove that quadrilaterals $ABCD$ and $WXYZ$ have the same area. **Hints**: 552 85 187 296

Problem 6.38 (Online Math Open Fall 2013). Let ABC be a triangle with $AB = 13$, $AC = 25$, and $\tan A = \frac{3}{4}$. Denote the reflections of B, C across $\overline{AC}$, $\overline{AB}$ by D, E, respectively, and let O be the circumcenter of triangle ABC. Let P be a point such that $\triangle DPO \sim \triangle PEO$, and let X and Y be the midpoints of the major and minor arcs $\widehat{BC}$ of the circumcircle of triangle ABC. Find $PX \cdot PY$. **Hints**: 30 303 608 **Sol:** p.260

Proposition 6.39 (Tangent Addition). *Consider angles A, B, C in the open interval $(-90^\circ, 90^\circ)$.*

(a) Let $x = \tan A$, $y = \tan B$, $z = \tan C$. Prove that

$$\tan(A + B + C) = \frac{(x + y + z) - xyz}{1 - (xy + yz + zx)}$$

if $xy + yz + zx \neq 1$, and is undefined otherwise.

(b) Generalize to multiple variables. **Hints**: 32 650 408 589 **Sol:** p.261

Proposition 6.40 (Schiffler Point). *Let ABC be a triangle with incenter I. Prove that the Euler lines of triangles AIB, BIC, CIA, and ABC are concurrent (called the **Schiffler point** of ABC).* **Hints**: 547 586 332

Problem 6.41 (IMO 2009/2). Let ABC be a triangle with circumcenter O. The points P and Q are interior points of the sides $\overline{CA}$ and $\overline{AB}$, respectively. Let K, L, and M be the midpoints of the segments BP, CQ, and PQ, respectively, and let Γ be the circle passing through K, L, and M. Suppose that the line PQ is tangent to the circle Γ. Prove that $OP = OQ$. **Hints**: 50 72 357

Problem 6.42 (APMO 2010/4). Let ABC be an acute triangle with $AB > BC$ and $AC > BC$. Denote by O and H the circumcenter and orthocenter of ABC. Suppose that the circumcircle of triangle AHC intersects the line AB at M (other than A), and the circumcircle of triangle AHB intersects the line AC at N (other than A). Prove that the circumcenter of triangle MNH lies on line OH. **Hints**: 642 121 445 **Sol:** p.261

Problem 6.43 (Shortlist 2006/G9). Points A_1, B_1, C_1 are chosen on the sides BC, CA, AB of a triangle ABC respectively. The circumcircles of triangles AB_1C_1, BC_1A_1, CA_1B_1 intersect the circumcircle of triangle ABC again at points A_2, B_2, C_2 respectively ($A_2 \neq A$, $B_2 \neq B$, $C_2 \neq C$). Points A_3, B_3, C_3 are symmetric to A_1, B_1, C_1 with respect to the midpoints of the sides BC, CA, AB respectively. Prove that triangles $A_2B_2C_2$ and $A_3B_3C_3$ are similar. **Hints**: 509 210 167

Problem 6.44 (MOP 2006). Given a cyclic quadrilateral $ABCD$ with circumcenter O and a point P on the plane, let O_1, O_2, O_3, O_4 denote the circumcenters of triangles PAB, PBC, PCD, PDA respectively. Prove that the midpoints of segments O_1O_3, O_2O_4, and OP are collinear. **Hints**: 29 431 **Sol:** p.263

Problem 6.45 (Shortlist 1998/G6). Let $ABCDEF$ be a convex hexagon such that $\angle B + \angle D + \angle F = 360^\circ$ and

$$\frac{AB}{BC} \cdot \frac{CD}{DE} \cdot \frac{EF}{FA} = 1.$$

Prove that

$$\frac{BC}{CA} \cdot \frac{AE}{EF} \cdot \frac{FD}{DB} = 1.$$

Hints: 153 668 649 197 **Sol:** p.264

Problem 6.46 (ELMO Shortlist 2013). Let ABC be a triangle inscribed in circle ω, and let the medians from B and C intersect ω at D and E respectively. Let O_1 be the center of the circle through D tangent to $\overline{AC}$ at C, and let O_2 be the center of the circle through E tangent to $\overline{AB}$ at B. Prove that O_1, O_2, and the nine-point center of ABC are collinear.

Hints: 371 655 554 203

CHAPTER 7

Barycentric Coordinates

> I suppose it is tempting, if the only tool you have is a hammer, to treat everything as if it were a nail.
>
> Maslow's Hammer

We now present another technique, barycentric coordinates. At the time of writing, it is surprisingly unknown to most olympiad contestants and problem writers.

In this chapter, the area notation $[XYZ]$ refers to signed areas (see Section 5.1). That means that the area $[XYZ]$ is positive if the points X, Y, Z are oriented in counterclockwise order, and negative otherwise.

7.1 Definitions and First Theorems

Throughout this section we fix a nondegenerate triangle ABC, called the **reference triangle**. (This is much like selecting an origin and axes in a Cartesian coordinate system.) Each point P in the plane is assigned an ordered triple of real numbers $P = (x, y, z)$ such that

$$\vec{P} = x\vec{A} + y\vec{B} + z\vec{C} \quad \text{and} \quad x + y + z = 1.$$

These are called the **barycentric coordinates** of point P with respect to triangle ABC.

Barycentric coordinates are also sometimes called **areal coordinates** because if $P = (x, y, z)$, then the signed area $[PBC]$ is equal to $x[ABC]$, and so on. In other words, these coordinates can be viewed as

$$P = \left(\frac{[PBC]}{[ABC]}, \frac{[PCA]}{[BCA]}, \frac{[PAB]}{[CAB]}\right).$$

The areas are signed in order to permit the point P to lie outside the triangle. If $P = (x, y, z)$ and A lie on opposite sides of $\overline{BC}$, then the signed areas of $[PBC]$ and $[ABC]$ have opposite signs and $x < 0$. In particular, the point P lies in the interior of ABC if and only if $x, y, z > 0$.

Observe that $A = (1, 0, 0)$, $B = (0, 1, 0)$ and $C = (0, 0, 1)$. This is why barycentric coordinates are substantially more suited for standard triangle geometry problems; the vertices are both simple and symmetric.

The soul of barycentric coordinates derives from the following result, which we state without proof.

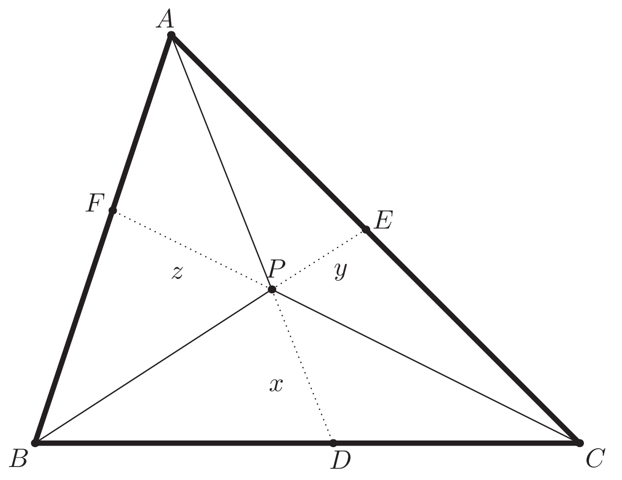

Figure 7.1A. Regions corresponding to the areas of ABC when P is inside the triangle.

Theorem 7.1 (Barycentric Area Formula). *Let P_1, P_2, P_3 be points with barycentric coordinates $P_i = (x_i, y_i, z_i)$ for $i = 1, 2, 3$. Then the signed area of $\triangle P_1P_2P_3$ is given by the determinant*

$$\frac{[P_1P_2P_3]}{[ABC]} = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}.$$

Again, the area is signed, following the convention in Section 5.1.

As a corollary, we derive the equation of a line.

Theorem 7.2 (Equation of a Line). *The equation of a line takes the form $ux + vy + wz = 0$ where u, v, w are real numbers. The u, v, and w are unique up to scaling.*

Proof. The main idea is that three points are collinear if and only if the signed area of their "triangle" is zero. Suppose we wish to characterize the points $P = (x, y, z)$ lying on a line XY, where $X = (x_1, y_1, z_1)$ and $Y = (x_2, y_2, z_2)$. Using the above area formula with $[PAB] = 0$, we find this occurs precisely when

$$0 = (y_1z_2 - y_2z_1)x + (z_1x_2 - z_2x_1)y + (x_1y_2 - x_2y_1)z,$$

i.e., $0 = ux + vy + wz$ for some constants u, v, w. $\square$

In particular, the equation for the line AB is simply $z = 0$, by substituting $(1, 0, 0)$ and $(0, 1, 0)$ into $ux + vy + wz = 0$. In general, the formula for a cevian through A is of the form $vy + wz = 0$, by substituting the point $A = (1, 0, 0)$.

In fact, the above techniques are already sufficient to prove both Ceva's and Menelaus's theorem.

Example 7.3 (Ceva's Theorem). Let D, E, F be points in the interiors of sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ of a triangle ABC. Then the cevians $\overline{AD}$, $\overline{BE}$, $\overline{CF}$ are concurrent if and only if

$$\frac{BD}{DC}\frac{CE}{EA}\frac{AF}{FB} = 1.$$

Proof. Define

$$D = (0, d, 1-d)$$
$$E = (1-e, 0, e)$$
$$F = (f, 1-f, 0)$$

where d, e, f are real numbers strictly between 0 and 1.

Then the corresponding equations of lines are

$$\overline{AD} : dz = (1-d)y$$
$$\overline{BE} : ex = (1-e)z$$
$$\overline{CF} : fy = (1-f)x.$$

We wish to show there is a nontrivial solution to this system of equations (i.e., one other than (0, 0, 0)) if and only if $def = (1-d)(1-e)(1-f)$, which is evidently equivalent to the constraint $\frac{BD}{DC}\frac{CE}{EA}\frac{AF}{FB} = 1$.

First suppose that a nontrivial solution (x, y, z) exists. Notice that if any of x, y, z is zero, then the others must all be zero as well. So we may assume $xyz \neq 0$. Now taking the product and cancelling xyz yields $def = (1-d)(1-e)(1-f)$.

On the other hand, suppose the condition $def = (1-d)(1-e)(1-f)$ holds. We opportunistically pick x, y, z. Put $y_1 = d$ and $z_1 = 1-d$. Then we require

$$x_1 = \frac{1-e}{e}(1-d) = \frac{f}{1-f}d$$

and this is okay since $def = (1-d)(1-e)(1-f)$; hence we can set x_1 as above. Thus $x = x_1$, $y = y_1$, and $z = z_1$ is a solution to the equations above.

However, there is no reason to believe that $x_1 + y_1 + z_1 = 1$, so the triple we found earlier may not actually correspond to a point. (However, we at least know $x_1, y_1, z_1 > 0$.) This is not a big issue: we instead consider the triple

$$(x, y, z) = \left(\frac{x_1}{x_1+y_1+z_1}, \frac{y_1}{x_1+y_1+z_1}, \frac{z_1}{x_1+y_1+z_1}\right)$$

which still satisfies the conditions, but now has sum 1. Thus this triple corresponds to the desired point of concurrency. □

The last step in the above proof illustrates that barycentric coordinates are *homogeneous*. Let us make his idea explicit. Suppose (x, y, z) lies on a line

$$ux + vy + wz = 0.$$

Then so does the "triple", $(2x, 2y, 2z)$, $(1000x, 1000y, 1000z)$ or indeed any multiple. In light of this, we permit **unhomogenized barycentric coordinates** by writing $(x : y : z)$ as shorthand for the appropriate triple

$$(x : y : z) = \left(\frac{x}{x+y+z}, \frac{y}{x+y+z}, \frac{z}{x+y+z}\right)$$

whenever $x + y + z \neq 0$. Note the use of colons instead of commas. An equivalent definition is as follows: for any nonzero k, the points $(x : y : z)$ and $(kx : ky : kz)$ are considered the same, and $(x : y : z) = (x, y, z)$ when $x + y + z = 1$.

This shorthand is convenient because such coordinates may still be "plugged in" to the line formula, often saving computations. For example, we have the following convenient corollary.

Theorem 7.4 (Barycentric Cevian). *Let $P = (x_1 : y_1 : z_1)$ be any point other than A. Then the points on line AP (other than A) can be parametrized by*

$$(t : y_1 : z_1)$$

where $t \in \mathbb{R}$ and $t + y_1 + z_1 \neq 0$.

On the other hand, it makes no sense to put unhomogenized coordinates into, say, the area formula. For these purposes, our usual coordinates (x, y, z) with the restriction $x + y + z = 1$ will be called **homogenized barycentric coordinates** and delimited with colons.

Problems for this Section

Problem 7.5. Find the coordinates for the midpoint of $\overline{AB}$. **Hint**: 623

Lemma 7.6 (Barycentric Conjugates). *Let $P = (x : y : z)$ be a point with $x, y, z \neq 0$. Show that the isogonal conjugate of P is given by*

$$P^* = \left(\frac{a^2}{x} : \frac{b^2}{y} : \frac{c^2}{z}\right)$$

and the isotomic conjugate is given by

$$P^t = \left(\frac{1}{x} : \frac{1}{y} : \frac{1}{z}\right).$$

Hint: 419

7.2 Centers of the Triangle

In Table 7.1 we give explicit forms for several centers of the reference triangle. Remember that $(u : v : w)$ refers to the point with coordinates $(\frac{u}{u+v+w}, \frac{v}{u+v+w}, \frac{w}{u+v+w})$; that is, we are not normalizing the coordinates.

This is so important we say it twice: *the coordinates here are unhomogenized.*

Here G, I, H, O denote the usual centroid, incenter, orthocenter, and circumcenter, while I_A denotes the A-excenter and K denotes the symmedian point. Notice that O and H are not particularly nice in barycentric coordinates (as compared to in, say, complex numbers), but I and K are particularly elegant.

It is often more useful to convert the trigonometric forms of H and O into expressions entirely in terms of the side lengths by

$$O = (a^2 S_A : b^2 S_B : c^2 S_C)$$

and

$$H = (S_B S_C : S_C S_A : S_A S_B)$$

Table 7.1. *Barycentric Coordinates of the Centers of a Triangle.*

Point/Coordinates	Sketch of Proof
$G = (1 : 1 : 1)$	Trivial
$I = (a : b : c)$	Areal definition
$I_A = (-a : b : c)$, etc.	Areal definition
$K = (a^2 : b^2 : c^2)$	Isogonal conjugates
$H = (\tan A : \tan B : \tan C)$	Areal definition
$O = (\sin 2A : \sin 2B : \sin 2C)$	Areal definition

where we define

$$S_A = \frac{b^2 + c^2 - a^2}{2}, \quad S_B = \frac{c^2 + a^2 - b^2}{2}, \quad S_C = \frac{a^2 + b^2 - c^2}{2}.$$

In Section 7.6 we investigate further properties of these expressions which provide a more viable way of dealing with them.

Just to provide some intuition on why Table 7.1 and Theorem 7.4 are useful, here is a simple example.

Example 7.7. Find the barycentric coordinates for the intersection of the internal angle bisector from A and the symmedian from B.

Solution. Suppose the desired intersection point is $P = (x : y : z)$. It is the intersection of lines AI and BK. According to Theorem 7.4, because $I = (a : b : c)$ we deduce that $y : z = b : c$. Similarly, because $K = (a^2 : b^2 : c^2)$ we deduce that $x : z = a^2 : c^2$. It is now elementary to find a solution to this: take

$$P = (a^2 : bc : c^2). \qquad \square$$

Moral: Cevians are extremely good in barycentric coordinates. And do not be afraid to use the law of sines if you have angles instead of side ratios.

Problems for this Section

Problem 7.8. Using the areal definition, show that $I = (a : b : c)$. Deduce the angle bisector theorem. **Hint**: 605

Problem 7.9. Find the barycentric coordinates for the intersection of the symmedian from A and the median from B. **Hint**: 463

7.3 Collinearity, Concurrence, and Points at Infinity

Theorem 7.1 can often be applied to show that three points are collinear. Specifically, we have the following result.

Theorem 7.10 (Collinearity). *Consider points P_1, P_2, P_3 with $P_i = (x_i : y_i : z_i)$ for $i = 1, 2, 3$. The three points are collinear if and only if*

$$0 = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}.$$

Note the coordinates need not be homogenized! This saves much computation.

Proof. The signed area of P_1, P_2, P_3 is zero (i.e., the points are collinear) if and only if

$$0 = \begin{vmatrix} \frac{x_1}{x_1+y_1+z_1} & \frac{y_1}{x_1+y_1+z_1} & \frac{z_1}{x_1+y_1+z_1} \\ \frac{x_2}{x_2+y_2+z_2} & \frac{y_2}{x_2+y_2+z_2} & \frac{z_2}{x_2+y_2+z_2} \\ \frac{x_3}{x_3+y_3+z_3} & \frac{y_3}{x_3+y_3+z_3} & \frac{z_3}{x_3+y_3+z_3} \end{vmatrix} \cdot [ABC].$$

The right-hand side simplifies as

$$\frac{[ABC]}{\prod_{i=1}^{3}(x_i + y_i + z_i)} \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}.$$

Because $[ABC] \neq 0$ the conclusion follows. □

This can be restated in the following useful form.

Proposition 7.11. *The line through two points $P = (x_1 : y_1 : z_1)$ and $Q = (x_2 : y_2 : z_2)$ is given precisely by the formula*

$$0 = \begin{vmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}.$$

We often use this in combination with Theorem 7.4 in order to intersect a cevian with an arbitrary line through two points.

We also have a similar criterion for when three lines are concurrent. However, before proceeding, we make a remark about **points at infinity**. We earlier defined

$$(x : y : z) = \left(\frac{x}{x+y+z}, \frac{y}{x+y+z}, \frac{z}{x+y+z} \right)$$

whenever $x + y + z \neq 0$. What of the case $x + y + z = 0$?

Consider two parallel lines $u_1x + v_1y + w_1z = 0$ and $u_2x + v_2y + w_2z = 0$. Because they are parallel, we know that the system

$$\begin{aligned} 0 &= u_1x + v_1y + w_1z \\ 0 &= u_2x + v_2y + w_2z \\ 1 &= x + y + z \end{aligned}$$

has no solutions (x, y, z). This is only possible when

$$\begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

However, this implies that the system of equations

$$\begin{aligned} 0 &= u_1x + v_1y + w_1z \\ 0 &= u_2x + v_2y + w_2z \\ 0 &= x + y + z \end{aligned}$$

has a nontrivial solution! (Conversely, if the lines are not parallel, the determinant is nonzero, and hence there is exactly one solution, namely (0, 0, 0).)

In light of this, we make each of our lines just "a little longer" by adding one point to it, a *point at infinity*. It is a point $(x : y : z)$ satisfying the equation of the line and the additional condition $x + y + z = 0$. With this addition, every two lines intersect; the lines that were parallel before now correspond to lines that intersect at points at infinity. Points at infinity are defined more precisely at the start of Chapter 9.

Example 7.12. Find the point at infinity along the internal bisector of angle A.

Solution. The point at infinity is $(-(b + c) : b : c)$. After all, it lies on the equation of the angle bisector, and the sum of its coordinates is zero. □

Theorem 7.13 (Concurrence). *Consider three lines*

$$\ell_i : u_ix + v_iy + w_iz = 0$$

for $i = 1, 2, 3$. *They are concurrent or all parallel if and only if*

$$0 = \begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix}.$$

Proof. This is essentially linear algebra. Consider the system of equations

$$\begin{aligned} 0 &= u_1x + v_1y + w_1z \\ 0 &= u_2x + v_2y + w_2z \\ 0 &= u_3x + v_3y + w_3z. \end{aligned}$$

It always has a solution $(x, y, z) = (0, 0, 0)$ and other solutions exist if and only if the lines concur (possibly at a point at infinity), which occurs only when the determinant of the matrix is zero. □

7.4 Displacement Vectors

In this section, we develop the notion of distance and direction through the use of vectors. This gives us a distance formula, and hence a circle formula, as well as a formula for the distance between two lines.

The chief definition is as follows. A **displacement vector** of two (normalized) points $P = (p_1, p_2, p_3)$ and $Q = (q_1, q_2, q_3)$ is denoted by $\overrightarrow{PQ}$ and is equal to $(q_1 - p_1, q_2 - p_2, q_3 - p_3)$. Note that the sum of the coordinates of a displacement vector is 0.

This section frequently involves translating the circumcenter O to the zero vector $\vec{0}$; this lets us invoke properties of the dot product described in Appendix A.3. This translation is valid since the point (x, y, z) satisfies $x + y + z = 1$, so the coordinates of the points do not change as a result; to be explicit, we can write

$$\vec{P} - \vec{O} = x\left(\vec{A} - \vec{O}\right) + y\left(\vec{B} - \vec{O}\right) + z\left(\vec{C} - \vec{O}\right)$$

since $x + y + z = 1$. As a result, however:

It is important that $x + y + z = 1$ when doing calculations with displacement vectors.

Our first major result is the distance formula.

Theorem 7.14 (Distance Formula). *Let P and Q be two arbitrary points and consider a displacement vector $\overrightarrow{PQ} = (x, y, z)$. Then the distance from P to Q is given by*

$$|PQ|^2 = -a^2yz - b^2zx - c^2xy.$$

Proof. Translate the coordinate plane so that the circumcenter O becomes the zero vector. Recall (from Appendix A.3) that this implies

$$\vec{A} \cdot \vec{A} = R^2 \text{ and } \vec{A} \cdot \vec{B} = R^2 - \frac{1}{2}c^2.$$

Here R is the circumradius of triangle ABC, as usual. Then we simply compute

$$|PQ|^2 = \left(x\vec{A} + y\vec{B} + z\vec{C}\right) \cdot \left(x\vec{A} + y\vec{B} + z\vec{C}\right).$$

Applying the properties of the dot product and using cyclic sum notation (defined in Section 0.3),

$$\begin{aligned}|PQ|^2 &= \sum_{\text{cyc}} x^2\vec{A} \cdot \vec{A} + 2\sum_{\text{cyc}} xy\vec{A} \cdot \vec{B} \\ &= R^2(x^2 + y^2 + z^2) + 2\sum_{\text{cyc}} xy\left(R^2 - \frac{1}{2}c^2\right).\end{aligned}$$

Collecting the R^2 terms,

$$\begin{aligned}|PQ|^2 &= R^2(x^2 + y^2 + z^2 + 2xy + 2yz + 2zx) - (c^2xy + a^2yz + b^2zx) \\ &= R^2(x + y + z)^2 - a^2yz - b^2zx - c^2xy \\ &= -a^2yz - b^2zx - c^2xy\end{aligned}$$

since $x + y + z = 0$, being the sum of the coordinates in a displacement vector. $\square$

As a consequence we can deduce the formula for the equation of a circle. It looks unwieldy, but it can often be tamed; see the remarks that follow the proof.

Theorem 7.15 (Barycentric Circle). *The general equation of a circle is*

$$-a^2yz - b^2zx - c^2xy + (ux + vy + wz)(x + y + z) = 0$$

for reals u, v, w.

Proof. Assume the circle has center (j, k, l) and radius r. Then applying the distance formula, we see that the circle is given by

$$-a^2(y-k)(z-l) - b^2(z-l)(x-j) - c^2(x-j)(y-k) = r^2.$$

Expand everything, and collect terms to get

$$-a^2yz - b^2zx - c^2xy + C_1x + C_2y + C_3z = C$$

for some hideous constants C_i and C. Since $x + y + z = 1$, we can rewrite

$$-a^2yz - b^2zx - c^2xy + ux + vy + wz = 0$$

as

$$-a^2yz - b^2zx - c^2xy + (ux + vy + wz)(x + y + z) = 0$$

where $u = C_1 - C$, etc. □

While this may look complicated, it turns out that circles that pass through vertices and sides are often very nice. For example, consider what occurs if the circle passes through $A = (1, 0, 0)$. The terms a^2yz, b^2zx, c^2xy all vanish, and accordingly we arrive at $u = 0$. Even if only one coordinate is zero, we still find many vanishing terms. Several examples are illustrated in the exercises.

As a result, whenever you are trying to solve a problem involving circumcircles through barycentrics, you should strive to set up the coordinates so that points on the circle are points on the sides, or better yet, vertices of the reference triangle. In other words, the choice of reference triangle is of paramount importance whenever circles appear.

Our last development for this section is a criterion to determine when two displacement vectors are perpendicular.

Theorem 7.16 (Barycentric Perpendiculars). *Let $\overrightarrow{MN} = (x_1, y_1, z_1)$ and $\overrightarrow{PQ} = (x_2, y_2, z_2)$ be displacement vectors. Then $\overline{MN} \perp \overline{PQ}$ if and only if*

$$0 = a^2(z_1y_2 + y_1z_2) + b^2(x_1z_2 + z_1x_2) + c^2(y_1x_2 + x_1y_2).$$

The proof is essentially the same as before: shift $\vec{O}$ to the zero vector, and then expand the condition $\overrightarrow{MN} \cdot \overrightarrow{PQ} = 0$, which is equivalent to perpendicularity. We encourage you to prove the theorem yourself before reading the following proof.

Proof. Translate $\vec{O}$ to $\vec{0}$. It is necessary and sufficient that

$$\left(x_1\vec{A} + y_1\vec{B} + z_1\vec{C}\right) \cdot \left(x_2\vec{A} + y_2\vec{B} + z_2\vec{C}\right) = 0.$$

Expanding, this is just

$$\sum_{\text{cyc}}\left(x_1x_2\vec{A}\cdot\vec{A}\right)+\sum_{\text{cyc}}\left((x_1y_2+x_2y_1)\,\vec{A}\cdot\vec{B}\right)=0.$$

Taking advantage of the fact that $\vec{O}=0$, we may rewrite this as

$$0=\sum_{\text{cyc}}(x_1x_2R^2)+\sum_{\text{cyc}}(x_1y_2+x_2y_1)\left(R^2-\frac{c^2}{2}\right).$$

This rearranges as

$$\begin{aligned}R^2\left(\sum_{\text{cyc}}(x_1x_2)+\sum_{\text{cyc}}(x_1y_2+x_2y_1)\right)&=\frac{1}{2}\sum_{\text{cyc}}\left((x_1y_2+x_2y_1)(c^2)\right)\\ R^2(x_1+y_1+z_1)(x_2+y_2+z_2)&=\frac{1}{2}\sum_{\text{cyc}}\left((x_1y_2+x_2y_1)(c^2)\right).\end{aligned}$$

But we know that $x_1+y_1+z_1=x_2+y_2+z_2=0$ in a displacement vector, so this becomes

$$\begin{aligned}R^2\cdot 0\cdot 0&=\frac{1}{2}\sum_{\text{cyc}}\left((x_1y_2+x_2y_1)(c^2)\right)\\ 0&=\sum_{\text{cyc}}\left((x_1y_2+x_2y_1)(c^2)\right).\end{aligned}$$ □

Theorem 7.16 is particularly useful when one of the displacement vectors is a side of the triangle. Several applications are given in the exercises, and more are seen in the examples section.

Problems for this Section

Lemma 7.17 (Barycentric Circumcircle). *The circumcircle (ABC) of the reference triangle has equation*

$$a^2yz+b^2zx+c^2xy=0.$$

Hint: 688

Problem 7.18. Consider a displacement vector $\overrightarrow{PQ}=(x_1,y_1,z_1)$. Show that $\overline{PQ}\perp\overline{BC}$ if and only if

$$0=a^2(z_1-y_1)+x_1(c^2-b^2).$$

Lemma 7.19 (Barycentric Perpendicular Bisector). *The perpendicular bisector of $\overline{BC}$ has equation*

$$0=a^2(z-y)+x(c^2-b^2).$$

7.5 A Demonstration from the IMO Shortlist

Before proceeding to even more obscure theory, we take the time to discuss an illustrative example. Here is a problem from the IMO Shortlist of 2011.

Example 7.20 (Shortlist 2011/G6). Let ABC be a triangle with $AB = AC$ and let D be the midpoint of $\overline{AC}$. The angle bisector of $\angle BAC$ intersects the circle through D, B, and C at the point E inside triangle ABC. The line BD intersects the circle through A, E, and B in two points B and F. The lines AF and BE meet at a point I, and the lines CI and BD meet at a point K. Show that I is the incenter of triangle KAB.

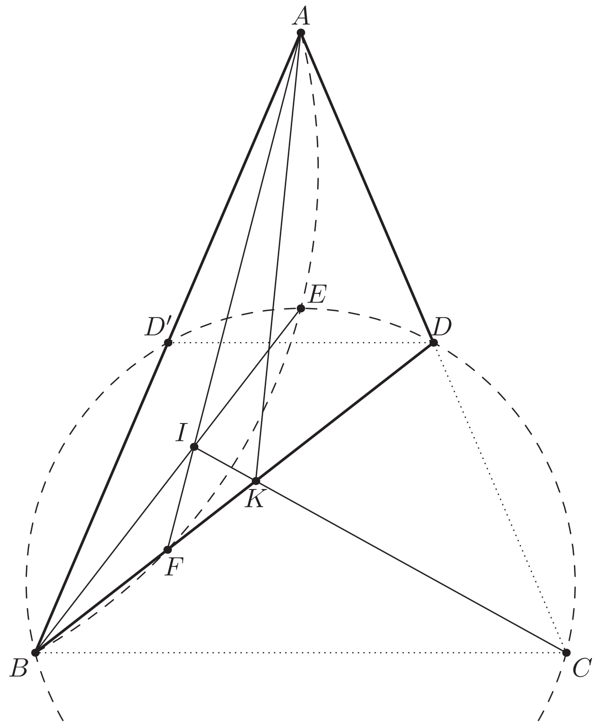

Figure 7.5A. IMO Shortlist 2011, Problem G6 (Example 7.20).

There are many nice and relatively painless synthetic observations that you can make in this problem. However, for the sake of discussion, we pretend we missed all of them. How should we apply barycentric coordinates?

Perhaps a better question is whether we should apply barycentric coordinates at all. There are two circles, but they seem relatively tame. There are lots of intersections of lines, but they seem to be mostly things that could be made into cevians. The final condition is about an angle bisector, which could pose difficulties, but we might make it.

A large part of this decision is based on what we choose for our reference triangle. At first we might be inclined to choose $\triangle ABC$, as the two circles in the problem pass through at least two vertices, and the condition $AB = AC$ is easy to encode. However, trying to prove that $\overline{BI}$ bisects $\angle ABD$, and that $\overline{AI}$ bisects $\angle BAK$, seems much less pleasant. Can we make at least one of them nicer?

That motivates a new choice of reference triangle: let us pick $\triangle ABD$ instead. That way, the $\overline{BE}$ bisection condition is extremely clean, and in fact almost immediate from the start (since E is the first point we compute). We still have the property that all circles pass through two vertices. Even better, the points F and K now lie on a side of the triangle, rather than just on some cevian (even though cevians are usually good too). And the second bisection condition looks much nicer now too, because we would only need to check $\frac{AB^2}{AK^2} = \frac{BF^2}{FK^2}$; since F and K lie on $\overline{BD}$, the right-hand side of this equality looks much better, and so the only truly nontrivial step would be computing AK^2. And finally, the isosceles condition is just $AB = 2AD$, which is trivial to encode.

It really is quite important that everything works out. A single thorn can doom the entire solution. We should always worry the most about the most time-consuming step of the entire plan—often this bottleneck takes longer to clear than the rest of the problem combined.

Let us begin. Set $A = (1, 0, 0)$, $B = (0, 1, 0)$, and $D = (0, 0, 1)$, and denote $a = BD$, $b = AD$, $c = AB = 2b$. We also abbreviate $\angle A = \angle BAD$, $\angle B = \angle DBA$, and $\angle D = \angle ADB$.

Our first objective is to compute E, so we need the equation of (BDC). We know that C is the reflection of A over D, and hence $C = (-1, 0, 2)$. Thus we are plugging in $B = (0, 1, 0)$, $C = (-1, 0, 2)$, and $D = (0, 0, 1)$ into the circle equation

$$(BDC) : -a^2yz - b^2zx - c^2xy + (x + y + z)(ux + vy + wz) = 0.$$

The points B and D now force $v = w = 0$—indeed this is why we want circles to pass through vertices. Now plugging in C gives

$$2b^2 - u = 0 \Rightarrow u = 2b^2.$$

Great. Now E lies on the bisector of $\angle BAD$. Hence, set $E = (t : 1 : 2)$ (which is equivalent to $(bs : b : 2b) = (bs : b : c)$, where $s = \frac{t}{b}$) for some t. We can now solve for t by just dropping it into the circle equation, which gives

$$-a^2(1)(2) - b^2(2)(t) - c^2(t)(1) + (3 + t)(2b^2 \cdot t) = 0.$$

Putting $c = 2b$, we enjoy a cancellation of all the t terms, leaving us with merely $2b^2 \cdot t^2 = 2a^2$, and hence $t = \pm\frac{a}{b}$. We pick $t > 0$ since E is in the interior, and accordingly we deduce $E = \left(\frac{a}{b} : 1 : 2\right)$, or

$$E = (a : b : 2b) = (a : b : c).$$

This means E is the incenter of $\triangle ABD$! Glancing back at the diagram, that implies that $\overline{BE}$ is the angle bisector of $\angle ABD$. And the explanation is simple: if D' is the reflection of D across $\overline{AE}$, then the arcs $D'E$ and DE of (BCD) are equal by simple symmetry. Hence $\angle D'BE = \angle EBD$. Oops. That was embarrassing. But let us trudge on.

The next step is to compute the point F. We first need the equation of (AEB). By proceeding as before with generic u, v, w, we may derive that $u = v = 0$ with the points

A and B. As for E, we require

$$-a^2bc - b^2ca - c^2ab + (a+b+c)(cw) = 0 \Rightarrow w = ab.$$

Now set $F = (0 : m : n)$ and throw this into our discovered circle formula. The computations give us

$$-a^2mn + (m+n)(abn) = 0 \Rightarrow -am + b(m+n) = 0$$

and so $m : n = b : a - b$. Hence

$$F = (0 : b : a - b) = \left(0 : \frac{b}{a} : \frac{a-b}{a}\right).$$

Wait, that is pretty clean. Why might that be?

Upon further thought, we see that

$$DF = \frac{b}{a} \cdot BD = b = AD.$$

In other words, F is the reflection of A over the bisector $\overline{ED}$. Is this obvious? Yes, it is—the center of (AEB) lies on $\overline{ED}$ by our ubiquitous Lemma 1.18. Cue sound of slap against forehead.

(At this point we might take a moment to verify that $a > b$, to rule out configuration issues. This just follows from the triangle inequality $a + b > 2b$.)

Next, we compute I. This is trivial, because $\overline{AF}$ and $\overline{BE}$ are cevians. Verify that

$$I = (a(a-b) : bc : c(a-b)) = \big(a(a-b) : 2b^2 : 2b(a-b)\big)$$

is the correct point.

We now wish to compute K. Let us set $K = (0 : y : z)$ and solve again for $y : z$. Because the points I, K, and C are collinear, our collinearity criterion (Theorem 7.10) gives us

$$0 = \begin{vmatrix} 0 & y & z \\ -1 & 0 & 2 \\ a(a-b) & 2b^2 & 2b(a-b) \end{vmatrix}.$$

Let us see if we make more zeros. Add $a(a-b)$ times the second row to the last to obtain

$$0 = 2\begin{vmatrix} 0 & y & z \\ -1 & 0 & 2 \\ 0 & b^2 & (b+a)(a-b) \end{vmatrix}.$$

Here we have factored the naturally occurring 2 in the bottom row. Apparently this implies, upon evaluating by minors (in the first column) that we have

$$0 = \begin{vmatrix} y & z \\ b^2 & a^2 - b^2 \end{vmatrix}.$$

Hence we discover $K = \left(0 : b^2 : a^2 - b^2\right) = \left(0, \frac{b^2}{a^2}, \frac{a^2-b^2}{a^2}\right)$. This is really nice as well. Actually, it implies in a similar way as before that

$$DK = \frac{b^2}{a} = \frac{AD^2}{BD} \Rightarrow DB \cdot DK = AD^2.$$

Did we miss another synthetic observation? This new discovery implies $\triangle DAK \sim \triangle DBA$, and hence $\angle KAD = \angle KBA$. That would mean $\angle BAK = \angle A - \angle B$, which is positive by $a > b$.

Our calculations have given us $\angle BAK = \angle A - \angle B$, meaning it suffices to prove that $\angle BAF = \frac{1}{2}(\angle A - \angle B)$. And yet $\angle BAE = \frac{1}{2}\angle A$, so we only need to prove $\angle FAE = \frac{1}{2}\angle B$. In a blinding flash of obvious, $\angle FAE = \angle FBE = \frac{1}{2}\angle B$ and we are done.

The calculation of K from F encodes all of the nontrivial synthetic steps of the problem, and our surprise at the resulting K led us naturally to the end. We write this up nicely, hiding the fact that we ever missed such steps.

Solution to Example 7.20. Let D' be the midpoint of $\overline{AB}$. Evidently the points B, D', D, E, C are concyclic. By symmetry, $DE = D'E$, and hence $\overline{BE}$ is a bisector of $\angle D'BD$. It follows that E is the incenter of triangle ABD. Since the center of (AEB) lies on ray DE by Lemma 1.18, it follows that the reflection of A over $\overline{ED}$ lies on (AEB), and hence is F.

We now claim that $DK \cdot DB = DA^2$. The proof is by barycentric coordinates on $\triangle ABD$. Set $A = (1,0,0)$, $B = (0,1,0)$, $C = (0,0,1)$ and let $a = BD$, $b = AD$, and $c = AB = 2b$. The observations above imply that $F = (0 : b : b-a)$ and $E = (a : b : c)$. This implies

$$I = (a(a-b) : bc : c(a-b)) = \left(a(a-b) : 2b^2 : 2b(a-b)\right).$$

Finally, $C = (-1, 0, 2)$. Hence if $K = (0 : y : z)$ then we have

$$0 = \begin{vmatrix} 0 & y & z \\ -1 & 0 & 2 \\ a(a-b) & 2b^2 & 2b(a-b) \end{vmatrix} = \begin{vmatrix} 0 & y & z \\ -1 & 0 & 2 \\ 0 & 2b^2 & 2(a^2-b^2) \end{vmatrix}$$

so $y : z = b^2 : (a^2 - b^2)$, so $K = \left(0, \frac{b^2}{a^2}, 1 - \frac{b^2}{a^2}\right)$. It follows immediately that $DK = \frac{b^2}{a}$ as desired.

Now remark that

$$DK \cdot DB = DA^2 \Rightarrow \triangle DAK \sim \triangle DBA \Rightarrow \angle FAD = \angle B.$$

So $\angle BAK = \angle A - \angle B$. But $\angle EAD = \frac{1}{2}\angle A$ and $\angle FAE = \angle FBE = \frac{1}{2}\angle B$ imply $\angle BAF = \frac{1}{2}(\angle A - \angle B)$, and we are done. $\square$

7.6 Conway's Notations

We now adapt **Conway's notation*** and define

$$S_A = \frac{b^2 + c^2 - a^2}{2}$$

and S_B and S_C analogously. Furthermore, let us define the shorthand $S_{BC} = S_B S_C$, and so on.

We first encountered these when we gave the coordinates of the circumcenter, and claimed they were friendlier than they seemed. This is because they happen to satisfy a

* The notation is named after John Horton Conway, a British mathematician.

lot of nice identities. For example, it is easy to see that $S_B + S_C = a^2$. Here are some less obvious ones.

Proposition 7.21 (Conway Identities). *Let S denote twice the area of triangle ABC. Then*

$$\begin{aligned} S^2 &= S_{AB} + S_{BC} + S_{CA} \\ &= S_{BC} + a^2 S_A \\ &= \frac{1}{2}(a^2 S_A + b^2 S_B + c^2 S_C) \\ &= (bc)^2 - S_A^2. \end{aligned}$$

In particular,

$$a^2 S_a + b^2 S_B - c^2 S_C = 2S_{AB}.$$

One might notice that there are a lot of $a^2 S_A$ and S_{AB} terms involved. This is because these are the coordinates of the circumcenter and orthocenter—hence these terms tend to arise naturally, and the identities provide a way of manipulating them.

More generally, if S is again equal to twice the area of triangle ABC, we define

$$S_\theta = S \cot \theta.$$

Here the angle is directed modulo 180°. The special case when $\theta = \angle A$ yields $S_A = \frac{1}{2}(b^2 + c^2 - a^2)$.

With this notation, we also have the following occasionally useful result.

Theorem 7.22 (Conway's Formula). *Let P be an arbitrary point. If $\beta = \measuredangle PBC$ and $\gamma = \measuredangle BCP$, then*

$$P = \left(-a^2 : S_C + S_\gamma : S_B + S_\beta\right).$$

The proof follows by computing the signed areas of triangles PBC, PAB, PCA and performing some manipulations. The proof is not particularly insightful and left to a diligent reader as an exercise. An example of an application appears in the exercises, Problem 7.37.

7.7 Displacement Vectors, Continued

In this section we refine some of our work in Section 7.4.

First of all, we look at our circle again:

$$-a^2yz - b^2zx - c^2xy + (x+y+z)(ux+vy+wz) = 0.$$

It might have seemed odd to insist on the negative signs in the first three terms, since we could have just as easily inverted the signs of u, v, w. It turns out that there is a good reason for this. Recall that we derived the circle formula by writing

$$(\text{distance from } (x, y, z) \text{ to center})^2 - \text{radius}^2 = 0.$$

This should look familiar! What happens if we substitute an arbitrary point (x, y, z) into the formula? In that case we obtain the *power* of a point with respect to the circle. Explicitly, we obtain the following lemma.

Lemma 7.23 (Barycentric Power of a Point). *Let ω be the circle given by*

$$-a^2yz - b^2zx - c^2xy + (x + y + z)(ux + vy + wz) = 0.$$

Then let $P = (x, y, z)$ be any point. Then

$$\text{Pow}_\omega(P) = -a^2yz - b^2zx - c^2xy + (x + y + z)(ux + vy + wz).$$

Note that we must have (x, y, z) homogenized here. Otherwise the distance formula breaks, and hence so does this lemma.

An easy but nonetheless indispensable consequence of Lemma 7.23 is the following lemma which gives us the radical axis of two circles.

Lemma 7.24 (Barycentric Radical Axis). *Suppose two non-concentric circles are given by the equations*

$$-a^2yz - b^2zx - c^2xy + (x + y + z)(u_1x + v_1y + w_1z) = 0$$
$$-a^2yz - b^2zx - c^2xy + (x + y + z)(u_2x + v_2y + w_2z) = 0.$$

Then their radical axis is given by

$$(u_1 - u_2)x + (v_1 - v_2)y + (w_1 - w_2)z = 0.$$

Proof. Just set the powers equal to each other and remark $x + y + z \neq 0$. Notice that this equation is homogeneous. □

We may also improve upon Theorem 7.16. In our proof of the theorem, we shifted $\vec{O}$ to zero and then used that

$$R^2(x_1 + y_1 + z_1)(x_2 + y_2 + z_2) = R^2 \cdot 0 \cdot 0 = 0.$$

In fact, we only need one of the displacement vectors to be zero for the entire product to be zero. For the other, we can get away with using a pseudo displacement vector; that is, we may cheat and, for example, write

$$\overrightarrow{HO} = \vec{H} - \vec{O} = \vec{H} = \vec{A} + \vec{B} + \vec{C} = (1, 1, 1)\,.$$

(Again, $\vec{O} = 0$ here. The lemma that $\vec{H} = \vec{A} + \vec{B} + \vec{C}$ under these conditions was proved in Chapter 6.)

Of course this is strictly nonsense, but the idea is there. Here is the formal statement.

Theorem 7.25 (Generalized Perpendicularity). *Suppose M, N, P, and Q are points with*

$$\overrightarrow{MN} = x_1\overrightarrow{AO} + y_1\overrightarrow{BO} + z_1\overrightarrow{CO}$$
$$\overrightarrow{PQ} = x_2\overrightarrow{AO} + y_2\overrightarrow{BO} + z_2\overrightarrow{CO}$$

such that either $x_1 + y_1 + z_1 = 0$ or $x_2 + y_2 + z_2 = 0$.

In that case, lines MN and PQ are perpendicular if and only if

$$0 = a^2(z_1y_2 + y_1z_2) + b^2(x_1z_2 + z_1x_2) + c^2(y_1x_2 + x_1y_2).$$

Proof. Repeat the proof of Theorem 7.16. □

This becomes useful when O or H is involved in a perpendicularity. For example, we can obtain the following corollary by finding the perpendicular line to $\overline{AO}$ through A.

Example 7.26. The tangent to (ABC) at A is given by

$$b^2z + c^2y = 0.$$

Proof. Let $P = (x, y, z)$ be a point on the tangent and assume as usual that $\vec{O} = 0$. The displacement vector $\overrightarrow{PA}$ is

$$\overrightarrow{PA} = (x - 1, y, z) = (x - 1)\vec{A} + y\vec{B} + z\vec{C}.$$

We can also use the pseudo displacement vector

$$\overrightarrow{AO} = \vec{A} - \vec{O} = 1\vec{A} + 0\vec{B} + 0\vec{C}.$$

Putting $(x_1, y_1, z_1) = (x - 1, y, z)$ and $(x_2, y_2, z_2) = (1, 0, 0)$ yields the result. □

7.8 More Examples

Our first example is the famous Pascal's theorem from projective geometry.

Example 7.27 (Pascal's Theorem). Let A, B, C, D, E, F be six distinct points on a circle Γ. Prove that the three intersections of lines AB and DE, BC and EF, and CD and FA are collinear.

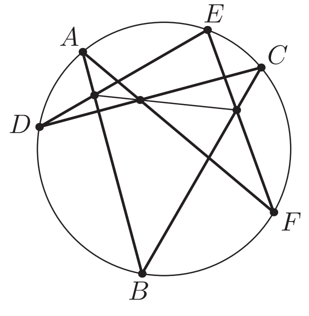

Figure 7.8A. Pascal's theorem (or one case thereof).

This problem seems okay because we have lots of intersections and only one circle.

Now we need to decide on a reference triangle. We might be tempted to pick ABC, but doing so loses much of the symmetry in the statement of Pascal's theorem. In addition, the lines DE and EF would fail to be cevians. Let us set reference triangle ACE instead—this way, our computations are symmetric, and the lines AB, DE, BC, EF, CD, FA are symmetric.

We can now proceed with the computation.

Solution. In some terrible notation, let $a = CE$, $b = EA$, $c = AE$. Set $A = (1, 0, 0)$, $C = (0, 1, 0)$, $E = (0, 0, 1)$. We still have to deal with the other points, which have a lot of freedom. Now we write

$$B = (x_1 : y_1 : z_1)$$
$$D = (x_2 : y_2 : z_2)$$
$$F = (x_3 : y_3 : z_3)$$

and hope for the best. Here, the points are subject to the constraint that they must lie on (ACE). That is, we have that

$$-a^2 y_i z_i - b^2 z_i x_i - c^2 x_i y_i = 0, \qquad i = 1, 2, 3.$$

Hopefully this will be helpful later, but for now there is no clear way to use this.

Now to actually compute the intersections. First, we need to smash the cevians AB and ED together. (For organization, I am always writing the vertex of the reference triangle first.) The line AB is the locus of points $(x : y : z)$ with $y : z = y_1 : z_1$, while the line ED is the locus of points with $x : y = x_2 : y_2$. Hence, the intersection of lines AB and ED is

$$\overline{AB} \cap \overline{ED} = \left(\frac{x_2}{y_2} : 1 : \frac{z_1}{y_1}\right).$$

(Here we are borrowing the intersection notation from Chapter 9, a bit prematurely. Bear with me.) We can do the exact same procedure to determine the other intersections:

$$\overline{CD} \cap \overline{AF} = \left(\frac{x_2}{z_2} : \frac{y_3}{z_3} : 1\right)$$

$$\overline{EF} \cap \overline{CB} = \left(1 : \frac{y_3}{x_3} : \frac{z_1}{x_1}\right).$$

Now to show that these are collinear, it suffices to show that the determinant

$$\begin{vmatrix} 1 & \frac{y_3}{x_3} & \frac{z_1}{x_1} \\ \frac{x_2}{y_2} & 1 & \frac{z_1}{y_1} \\ \frac{x_2}{z_2} & \frac{y_3}{z_3} & 1 \end{vmatrix}$$

is zero. (We have lined up the 1s on the main diagonal.) Seeing this, we are inspired to rewrite our given condition as

$$a^2 \cdot \frac{1}{x_1} + b^2 \cdot \frac{1}{y_1} + c^2 \cdot \frac{1}{z_1} = 0$$
$$a^2 \cdot \frac{1}{x_2} + b^2 \cdot \frac{1}{y_2} + c^2 \cdot \frac{1}{z_2} = 0$$
$$a^2 \cdot \frac{1}{x_3} + b^2 \cdot \frac{1}{y_3} + c^2 \cdot \frac{1}{z_3} = 0.$$

Linear algebra now tells us that

$$0 = \begin{vmatrix} \frac{1}{x_1} & \frac{1}{y_1} & \frac{1}{z_1} \\ \frac{1}{x_2} & \frac{1}{y_2} & \frac{1}{z_2} \\ \frac{1}{x_3} & \frac{1}{y_3} & \frac{1}{z_3} \end{vmatrix}$$

but this equals

$$\frac{1}{x_2 y_3 z_1} \cdot \begin{vmatrix} \frac{z_1}{x_1} & \frac{z_1}{y_1} & 1 \\ 1 & \frac{x_2}{y_2} & \frac{x_2}{z_2} \\ \frac{y_3}{x_3} & 1 & \frac{y_3}{z_3} \end{vmatrix}$$

which quickly implies that the first determinant is zero. $\square$

There is actually little geometry involved in our proof of Pascal's theorem. In fact, there is very little special about the use of barycentric coordinates versus any other type of symmetric coordinates. Indeed they are a special case of **homogeneous coordinates**, i.e., a coordinate system that identifies $(kx : ky : kz)$ with (x, y, z). This is why the determinant calculations involved virtually no geometric observations.

Our next example involves a pair of incircles.

Example 7.28. Let ABC be a triangle and D a point on $\overline{BC}$. Let I_1 and I_2 denote the incenters of triangles ABD and ACD, respectively. Lines BI_2 and CI_1 meet at K. Prove that K lies on $\overline{AD}$ if and only if $\overline{AD}$ is the angle bisector of angle A.

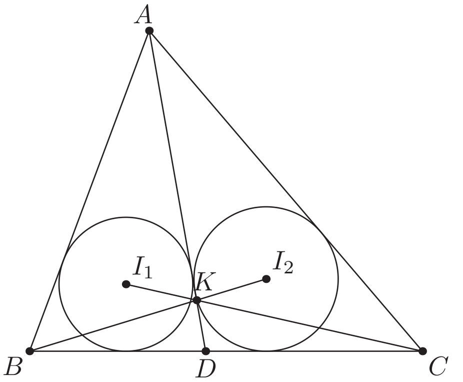

Figure 7.8B. Using barycentric coordinates to tame incircles.

The first thing we notice in this problem is the incenters. This should evoke fear, because we do not know much about how to deal with incenters other than that of ABC. Fortunately, these ones seem somewhat bound to ABC, so we might be okay.

We take ABC as the reference triangle. (After all, we do have a set of concurrent cevians, so this seems like something we want to use.) Now the hard part is deciding how to determine I_2.

Perhaps we can phrase I_2 as the intersection of two angle bisectors. Obviously one of them is the C-bisector. For the other, we consider the bisector $\overline{DI_2}$ (using $\overline{AI_2}$ will also work). If we can intersect the lines DI_2 and CI_2, this will of course give I_2.

So how can we handle $\overline{DI_2}$? If we let C_1 be the intersection of $\overline{DI_2}$ with $\overline{AC}$, then C_1 splits side $\overline{AC}$ in an $AD : AC$ ratio, by the angle bisector theorem. This suggests setting $d = AD$, $p = CD$, $q = BD$, where $p + q = a$. In that case, $C_1 = (p : 0 : d)$.

One might pause to worry about the fact we now have six variables. There are some relations, $p + q = a$ and Stewart's theorem, but we prefer not to use these. The reassurance is that so far all our equations have been of linear degree, so high degrees seem unlikely to appear. Indeed, we see that the solution is very short.

Solution to Example 7.28. Use barycentric coordinates with respect to ABC. Put $AD = d$, $CD = p$, $BD = q$.

Let ray DI_2 meet $\overline{AC}$ at C_1. Evidently $C_1 = (p : 0 : d)$ while $D = (0 : p : q)$.

Thus if $I_2 = (a : b : t)$ then we have

$$\begin{vmatrix} p & 0 & d \\ 0 & p & q \\ a & b & t \end{vmatrix} = 0 \Rightarrow t = \frac{ad + bq}{p}$$

which yields

$$I_2 = (ap : bp : ad + bq).$$

Similarly,

$$I_1 = (aq : ad + cp : cq).$$

So lines BI_2 and CI_1 intersect at a point

$$K = (apq : p(ad + cp) : q(ad + bq)).$$

This lies on line AD, so

$$\frac{p}{q} = \frac{p(ad + cp)}{q(ad + bq)}.$$

Hence we obtain $cp = bq$ or $p : q = b : c$ implying D is the foot of the angle bisector. $\square$

Next in line is a problem from the USAMO in 2008.

Example 7.29 (USAMO 2008/2). Let ABC be an acute, scalene triangle, and let M, N, and P be the midpoints of $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, respectively. Let the perpendicular bisectors of $\overline{AB}$ and $\overline{AC}$ intersect ray AM in points D and E respectively, and let lines BD and CE intersect in point F, inside triangle ABC. Prove that points A, N, F, and P all lie on one circle.

This one is actually a straightforward computation (but not a straightforward synthetic problem) with reference triangle ABC, but we have selected it to illustrate the use of

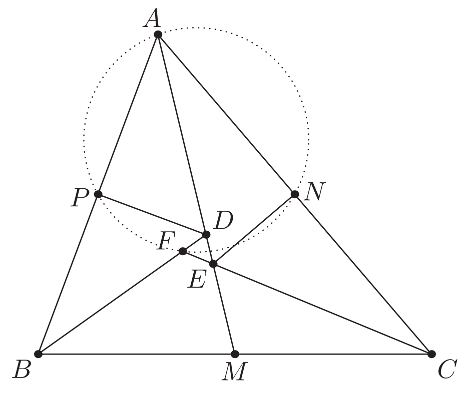

Figure 7.8C. Show that A, N, F, P are concyclic.

determinants and Conway's notation. There are only two nontrivial steps we will make. The first is to compute D as the intersection of lines PO and AM (where O is of course the circumcenter); there are other approaches but this is (I think) the cleanest. The second is that a homothety with ratio 2 at A to check that F lies on (ANP); we show that the reflection of A over F lies on (ABC), which solves the problem. All else is algebra.

Solution to Example 7.29. First, we find the coordinates of D. As D lies on $\overline{AM}$, we know $D = (t : 1 : 1)$ for some t. Now by Lemma 7.19, we find

$$0 = b^2(t-1) + (a^2 - c^2) \Rightarrow t = \frac{c^2 + b^2 - a^2}{b^2}.$$

Thus we obtain

$$D = \left(2S_A : c^2 : c^2\right).$$

Analogously $E = (2S_A : b^2 : b^2)$, and it follows that

$$F = \left(2S_A : b^2 : c^2\right).$$

The sum of the coordinates of F is

$$(b^2 + c^2 - a^2) + b^2 + c^2 = 2b^2 + 2c^2 - a^2.$$

Hence the reflection of A over F is simply

$$2F - A = \left(-a^2 : 2b^2 : 2c^2\right).$$

It is evident that F' lies on $(ABC) : -a^2yz - b^2zx - c^2xy = 0$, and we are done. $\square$

Our final example is the closing problem from Chapter 3. It stretches the power of our technique by showing even intersections with circles can be handled.

Example 7.30 (USA TSTST 2011/4). Acute triangle ABC is inscribed in circle ω. Let H and O denote its orthocenter and circumcenter, respectively. Let M and N be the midpoints of sides AB and AC, respectively. Rays MH and NH meet ω at P and Q, respectively. Lines MN and PQ meet at R. Prove that $\overline{OA} \perp \overline{RA}$.

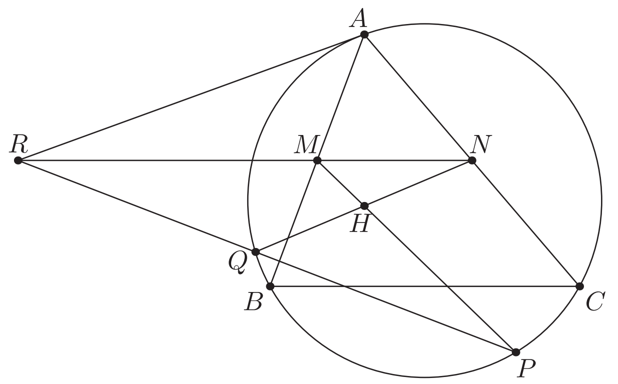

Figure 7.8D. Show that $\overline{RA}$ is a tangent.

This one is going to be wilder. We step back and plan before we begin the siege.

Intersecting $\overline{MN}$ and $\overline{PQ}$, and then showing the result is tangent, does not seem too hard. We have M, N, and H for free. However, it seems trickier to obtain the coordinates of P and Q.

Not all hope is lost. We want to avoid solving quadratics, so consider what happens when we intersect line MH with circle (ABC). Because $M = (1 : 1 : 0)$ and $H = (S_{BC} : S_{CA} : S_{AB})$, the equation of line MH can be computed as

$$0 = x - y + \left(\frac{S_{AC} - S_{BC}}{S_{AB}}\right) z.$$

Also, we of course know $0 = a^2yz + b^2zx + c^2xy$. Let us select $P = (x : y : -S_{AB})$. Then our system of equations in x and y is

$$\begin{aligned} x + y &= S_C\left(S_A - S_B\right) \\ c^2xy &= S_AS_B\left(a^2y + b^2x\right). \end{aligned}$$

We can attempt to solve directly for x, and we get some sloppy quadratic of the form $\alpha x^2 + \beta x + \gamma = 0$ for some (messy) expressions α, β, γ. The quadratic formula seems hopeless at this point.

But we are not stuck yet. Think about the two values of x. They correspond to the coordinates of two points, P and second point P', which has been marked in Figure 7.8E.

But the point P' is very familiar—it is just the point diametrically opposite C, and also the reflection of H over M. So it is straightforward to compute the value of x corresponding to P'. Vieta's formulas then tell us the sum of the roots of our quadratic is $-\frac{\beta}{\alpha}$, and we get our value of x for free.

Now we can start the computation.

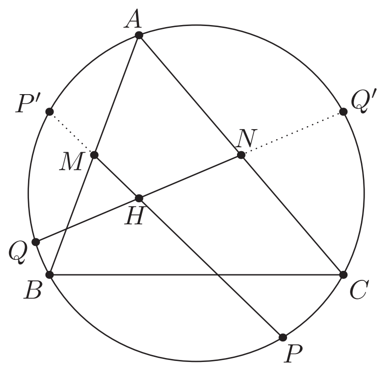

Figure 7.8E. Vieta jumping, anyone?

Solution to Example 7.30. We use barycentrics on ABC.

First, we compute the coordinates of P', the second intersection of line MH with (ABC). Since it is the reflection of $H = (S_{BC}, S_{CA}, S_{AB})$ over M, and the coordinates of H sum to $S_{AB} + S_{BC} + S_{CA}$, we may write

$$\begin{aligned} P' &= 2\left(\frac{S_{AB}+S_{BC}+S_{CA}}{2} : \frac{S_{AB}+S_{BC}+S_{CA}}{2} : 0\right) \\ &\quad - (S_{BC} : S_{CA} : S_{AB}) \\ &= (S_{AB}+S_{AC} : S_{AB}+S_{BC} : -S_{AB}) \\ &= \left(a^2 S_A : b^2 S_B : -S_{AB}\right). \end{aligned}$$

Now let us determine the coordinates of P, where we let $P = \left(x' : y' : z'\right) = \left(x' : y' : -S_{AB}\right)$ (valid since we just scale the coordinates so that $z' = -S_{AB}$). Because it lies on line MH, we find

$$0 = x' - y' + \left(\frac{S_{AC} - S_{BC}}{S_{AB}}\right) z' \Rightarrow y' = x' + S_{BC} - S_{AC}.$$

Also, we know that $a^2 y'z' + b^2 z'x' + c^2 x'y' = 0$, which gives

$$c^2 x'y' = S_{AB}\left(a^2 y' + b^2 x'\right).$$

Substituting, we have

$$c^2\left(x'\left(x' + S_{BC} - S_{AC}\right)\right) = S_{AB}\left(a^2\left(x' + S_{BC} - S_{AC}\right) + b^2 x'\right).$$

Collecting like terms gives the quadratic

$$c^2 x'^2 + \left[c^2\left(S_{BC} - S_{AC}\right) - (a^2+b^2) S_{AB}\right] x' + \text{constant} = 0.$$

By Vieta's formulas, then, the x' we seek is just

$$\frac{a^2+b^2}{c^2} S_{AB} - S_{BC} + S_{AC} - a^2 S_A.$$

Writing $a^2 = S_{AB} + S_{AC}$ in hopes of clearing out some terms, this becomes

$$\frac{a^2+b^2-c^2}{c^2} S_{AB} - S_{BC} = \frac{S_A S_B S_C}{c^2} - S_{BC}.$$

Now $y' = \frac{S_A S_B S_C}{c^2} - S_{AC}$. Cleaning further,

$$P = \left(S_B^2 S_C : S_A^2 S_C : c^2 S_{AB}\right).$$

Analogous calculations give that

$$Q = \left(S_B S_C^2 : b^2 S_{AC} : S_A^2 S_B\right).$$

Finding the equation of line PQ looks painful, so let us find where R should be first. Let the tangent to A meet line MN at R'. It is straightforward to derive that $R' = \left(b^2 - c^2 : b^2 : -c^2\right)$. Now we can just take a determinant. To show the three points P, Q, R' are collinear it suffices to check that

$$0 = \begin{vmatrix} S_B^2 S_C & S_A^2 S_C & c^2 S_A S_B \\ S_B S_C^2 & b^2 S_A S_C & S_A^2 S_B \\ b^2 - c^2 & b^2 & -c^2 \end{vmatrix}.$$

Note that $S_B^2 S_C - S_A^2 S_C - c^2 S_A S_B = c^2 \left[S_C(S_B - S_A) - S_A S_B\right]$. So upon subtracting the second and third columns from the first, this factors as

$$(S_{BC} - S_{AB} - S_{AC}) \cdot \begin{vmatrix} c^2 & S_A^2 S_C & c^2 S_A S_B \\ b^2 & b^2 S_A S_C & S_A^2 S_B \\ 0 & b^2 & -c^2 \end{vmatrix}.$$

To show this is zero, it suffices to check that

$$b^2 \left(c^2 S_A^2 S_B - b^2 c^2 S_A S_B\right) = c^2 \left(b^2 S_A^2 S_C - b^2 c^2 S_A S_C\right).$$

The left-hand side factors as $S_A S_B b^2 c^2 \left(S_A - b^2\right) = -S_A S_B S_C b^2 c^2$ and so does the right-hand side, so we are done. □

This is certainly a somewhat brutal solution, but the calculation can be carried out within a half hour (and two pages) with some experience (and little insight). Notice how Conway's notation kept the expressions manageable.

7.9 When (Not) to Use Barycentric Coordinates

To summarize, let us discuss briefly when barycentrics are useful.

- Cevians are wonderful in every way, shape, and form. Know them, use them, love them. Pick reference triangles in which many lines become cevians.
- Problems heavily involving centers of a prominent triangle are in general good, because we have nice forms for most of the centers.
- Intersections of lines, collinearity, and concurrence are fine. Bonus points when cevians are involved.

- Problems that are symmetric around the vertices of a triangle. Because barycentric coordinates are also symmetric, this allows us to take advantage of the nice symmetry, unlike with Cartesian coordinates.
- Ratios, lengths, or areas.
- Problems with few points. This is kind of obvious—the fewer points you have to compute, the better.

In contrast, here are things that barycentric coordinates do not handle well.

- Lots of circles. One can sometimes find a way around circles (for example, if only the radical axis or power of a point is relevant).
- Circles that do not pass through vertices of sides of a reference triangle. In general, the equation of a circle through three completely arbitrary points will be very ugly. However, the circle becomes much more tractable if the points it passes through have zeros.
- Arbitrary circumcenters.
- General angle conditions. Of course, there are exceptions; they typically involve angle conditions that can be translated into length conditions. The angle bisector theorem is your friend here.

7.10 Problems

There are quite a few contest problems that can be solved by barycentrics; this represents a rather small subset of problems I have encountered that are susceptible. Part of the reason is that, at the time of writing, barycentrics are a relatively unknown technique. As a result, test-writers are not aware when a problem they propose is trivialized by barycentric coordinates, as they would have been for a problem approachable by either complex numbers or Cartesian coordinates.

Lemma 7.31. *Let ABC be a triangle with altitude $\overline{AL}$ and let M be the midpoint of $\overline{AL}$. If K is the symmedian point of triangle ABC, prove that $\overline{KM}$ bisects $\overline{BC}$.* **Hints**: 652 393

Problem 7.32. Let I and G denote the incenter and centroid of a triangle ABC and let N denote the **Nagel point**; this is the intersection of the cevians that join A to the contact point of the A-excircle on $\overline{BC}$, and similarly for B and C. Prove that I, G, N are collinear and that $NG = 2GI$. **Hints**: 271 243

Problem 7.33 (IMO 2014/4). Let P and Q be on segment BC of an acute triangle ABC such that $\angle PAB = \angle BCA$ and $\angle CAQ = \angle ABC$. Let M and N be the points on AP and AQ, respectively, such that P is the midpoint of AM and Q is the midpoint of AN. Prove that the intersection of BM and CN is on the circumference of triangle ABC. **Hints**: 486 574 251 **Sol:** p.265

Problem 7.34 (EGMO 2013/1). The side BC of triangle ABC is extended beyond C to D so that $CD = BC$. The side CA is extended beyond A to E so that $AE = 2CA$. Prove that, if $AD = BE$, then triangle ABC is right-angled. **Hint**: 188 **Sol:** p.265

Problem 7.35 (ELMO Shortlist 2013). In $\triangle ABC$, a point D lies on line BC. The circumcircle of ABD meets $\overline{AC}$ at F (other than A), and the circumcircle of ADC meets

$\overline{AB}$ at E (other than A). Prove that as D varies, the circumcircle of AEF always passes through a fixed point other than A, and that this point lies on the median from A to BC. **Hints**: 657 653

Problem 7.36 (IMO 2012/1). Given triangle ABC the point J is the center of the excircle opposite the vertex A. This excircle is tangent to side BC at M, and to lines AB and AC at K and L, respectively. Lines LM and BJ meet at F, and lines KM and CJ meet at G. Let S be the point of intersection of lines AF and BC, and let T be the point of intersection of lines AG and BC. Prove that M is the midpoint of ST. **Hints**: 447 280 **Sol:** p.266

Problem 7.37 (Shortlist 2001/G1). Let A_1 be the center of the square inscribed in acute triangle ABC with two vertices of the square on side $\overline{BC}$. Thus one of the two remaining vertices of the square is on side $\overline{AB}$ and the other is on $\overline{AC}$. Points B_1, C_1 are defined in a similar way for inscribed squares with two vertices on sides $\overline{AC}$ and $\overline{AB}$, respectively. Prove that lines AA_1, BB_1, CC_1 are concurrent. **Hints**: 123 466

Problem 7.38 (USA TST 2008/7). Let ABC be a triangle with G as its centroid. Let P be a variable point on segment BC. Points Q and R lie on sides AC and AB respectively, such that $\overline{PQ} \parallel \overline{AB}$ and $\overline{PR} \parallel \overline{AC}$. Prove that, as P varies along segment BC, the circumcircle of triangle AQR passes through a fixed point X such that $\angle BAG = \angle CAX$. **Hints**: 6 647 **Sol:** p.266

Problem 7.39 (USAMO 2001/2). Let ABC be a triangle and let ω be its incircle. Denote by D_1 and E_1 the points where ω is tangent to sides BC and AC, respectively. Denote by D_2 and E_2 the points on sides BC and AC, respectively, such that $CD_2 = BD_1$ and $CE_2 = AE_1$, and denote by P the point of intersection of segments AD_2 and BE_2. Circle ω intersects segment AD_2 at two points, the closer of which to the vertex A is denoted by Q. Prove that $AQ = D_2P$. **Hints**: 320 160

Problem 7.40 (USA TSTST 2012/7). Triangle ABC is inscribed in circle Ω. The interior angle bisector of angle A intersects side BC and Ω at D and L (other than A), respectively. Let M be the midpoint of side BC. The circumcircle of triangle ADM intersects sides AB and AC again at Q and P (other than A), respectively. Let N be the midpoint of segment PQ, and let H be the foot of the perpendicular from L to line ND. Prove that line ML is tangent to the circumcircle of triangle HMN. **Hints**: 381 345 576

Problem 7.41. Let ABC be a triangle with incenter I. Let P and Q denote the reflections of B and C across $\overline{CI}$ and $\overline{BI}$, respectively. Show that $\overline{PQ} \perp \overline{OI}$, where O is the circumcenter of ABC. **Hints**: 396 461

Lemma 7.42. *Let ABC be a triangle with circumcircle Ω and let T_A denote the tangency points of the A-mixtilinear incircle to Ω. Define T_B and T_C similarly. Prove that lines AT_A, BT_B, CT_C, IO are concurrent, where I and O denote the incenter and circumcenter of triangle ABC.* **Hints**: 490 54 602 488 **Sol:** p.267

Problem 7.43 (USA December TST for IMO 2012). In acute triangle ABC, $\angle A < \angle B$ and $\angle A < \angle C$. Let P be a variable point on side BC. Points D and E lie on sides AB and AC, respectively, such that $BP = PD$ and $CP = PE$. Prove that as P moves along side

BC, the circumcircle of triangle ADE passes through a fixed point other than A. **Hints**: 179 144 137

Problem 7.44 (Sharygin 2013). Let C_1 be an arbitrary point on side AB of $\triangle ABC$. Points A_1 and B_1 are on rays BC and AC such that $\angle AC_1B_1 = \angle BC_1A_1 = \angle ACB$. The lines AA_1 and BB_1 meet in point C_2. Prove that all the lines C_1C_2 have a common point. **Hints**: 51 12 66 304 **Sol:** p.268

Problem 7.45 (APMO 2013/5). Let $ABCD$ be a quadrilateral inscribed in a circle ω, and let P be a point on the extension of AC such that $\overline{PB}$ and $\overline{PD}$ are tangent to ω. The tangent at C intersects $\overline{PD}$ at Q and the line AD at R. Let E be the second point of intersection between $\overline{AQ}$ and ω. Prove that B, E, R are collinear. **Hints**: 379 524 129

Problem 7.46 (USAMO 2005/3). Let ABC be an acute-angled triangle, and let P and Q be two points on its side BC. Construct a point C_1 in such a way that the convex quadrilateral $APBC_1$ is cyclic, $\overline{QC_1} \parallel \overline{CA}$, and C_1 and Q lie on opposite sides of line AB. Construct a point B_1 in such a way that the convex quadrilateral $APCB_1$ is cyclic, $\overline{QB_1} \parallel \overline{BA}$, and B_1 and Q lie on opposite sides of line AC. Prove that the points B_1, C_1, P, and Q lie on a circle. **Hints**: 191 325 204

Problem 7.47 (Shortlist 2011/G2). Let $A_1A_2A_3A_4$ be a non-cyclic quadrilateral. For $1 \le i \le 4$, let O_i and r_i be the circumcenter and the circumradius of triangle $A_{i+1}A_{i+2}A_{i+3}$ (where $A_{i+4} = A_i$). Prove that

$$\frac{1}{O_1A_1^2 - r_1^2} + \frac{1}{O_2A_2^2 - r_2^2} + \frac{1}{O_3A_3^2 - r_3^2} + \frac{1}{O_4A_4^2 - r_4^2} = 0.$$

Hints: 468 588 224 621 **Sol:** p.269

Problem 7.48 (Romania TST 2010). Let ABC be a scalene triangle, let I be its incenter, and let A_1, B_1, and C_1 be the points of contact of the excircles with the sides BC, CA, and AB, respectively. Prove that the circumcircles of the triangles AIA_1, BIB_1, and CIC_1 have a common point different from I. **Hints**: 549 23 94

Problem 7.49 (ELMO 2012/5). Let ABC be an acute triangle with $AB < AC$, and let D and E be points on side BC such that $BD = CE$ and D lies between B and E. Suppose there exists a point P inside ABC such that $\overline{PD} \parallel \overline{AE}$ and $\angle PAB = \angle EAC$. Prove that $\angle PBA = \angle PCA$. **Hints**: 171 229 **Sol:** p.270

Problem 7.50 (USA TST 2004/4). Let ABC be a triangle. Choose a point D in its interior. Let ω_1 be a circle passing through B and D and ω_2 be a circle passing through C and D so that the other point of intersection of the two circles lies on $\overline{AD}$. Let ω_1 and ω_2 intersect side BC at E and F, respectively. Denote by X the intersection of lines DF and AB, and let Y the intersection of DE and AC. Show that $\overline{XY} \parallel \overline{BC}$. **Hints**: 301 206 567 126

Problem 7.51 (USA TSTST 2012/2). Let $ABCD$ be a quadrilateral with $AC = BD$. Diagonals AC and BD meet at P. Let ω_1 and O_1 denote the circumcircle and the circumcenter of triangle ABP. Let ω_2 and O_2 denote the circumcircle and circumcenter of triangle CDP. Segment BC meets ω_1 and ω_2 again at S and T (other than B and C), respectively. Let

M and N be the midpoints of minor arcs $\widehat{SP}$ (not including B) and $\widehat{TP}$ (not including C). Prove that $\overline{MN} \parallel \overline{O_1O_2}$. **Hints**: 651 518 664 364

Problem 7.52 (IMO 2004/5). In a convex quadrilateral $ABCD$, the diagonal BD bisects neither the angle ABC nor the angle CDA. Point P lies inside $ABCD$ with $\angle PCB = \angle DBA$ and $\angle PDC = \angle BDA$. Prove that $ABCD$ is a cyclic quadrilateral if and only if $AP = CP$. **Hints**: 117 266 641 349 **Sol:** p.270

Problem 7.53 (Shortlist 2006/G4). Let ABC be a triangle with $\angle C < \angle A < 90^\circ$. Select point D on side AC so that $BD = BA$. The incircle of ABC is tangent to $\overline{AB}$ and $\overline{AC}$ at points K and L, respectively. Let J be the incenter of triangle BCD. Prove that the line KL bisects $\overline{AJ}$. **Hints**: 5 295 281 394

Part III
Farther from Kansas

CHAPTER 8

Inversion

Out of nothing I have created a strange new universe. — János Bolyai

In this chapter we discuss the method of inversion in the plane. This technique is useful for turning circles into lines and for handling tangent figures.

8.1 Circles are Lines

A **cline** (or **generalized circle**) refers to either a circle or a line. Throughout the chapter, we use "circle" and "line" to refer to the ordinary shapes, and "cline" when we wish to refer to both.

The idea is to view every line as a circle with infinite radius. We add a special point P_∞ to the plane, which every ordinary line passes through (and no circle passes through). This is called the **point at infinity**. Therefore, every choice of three distinct points determines a unique cline—three ordinary points determine a circle, while two ordinary points plus the point at infinity determine a line.

With this said, we can now define an inversion. Let ω be a circle with center O and radius R. We say an **inversion** about ω is a map (that is, a transformation) which does the following.

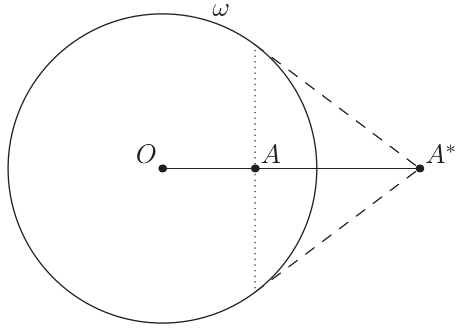

Figure 8.1A. A^* is the image of the point A when we take an inversion about ω.

- The center O of the circle is sent to P_∞.
- The point P_∞ is sent to O.

- For any other point A, we send A to the point A^* lying on ray OA such that $OA \cdot OA^* = r^2$.

Try to apply the third rule to $A = O$ and $A = P_\infty$, and the motivation for the first two rules becomes much clearer. The way to remember it is "$\frac{r^2}{0} = \infty$" and "$\frac{r^2}{\infty} = 0$".

At first, this rule seems arbitrary and contrived. What good could it do? First, we make a few simple observations.

1. A point A lies on ω if and only if $A = A^*$. In other words, the points of ω are fixed.
2. Inversion swaps pairs of points. In other words, the inverse of A^* is A itself. In still other words, $(A^*)^* = A$.

We can also find a geometric interpretation for this mapping, which provides an important setting in which inverses arise naturally.

Lemma 8.1 (Inversion and Tangents). *Let A be a point inside ω, other than O, and A^* be its inverse. Then the tangents from A^* to ω are collinear with A.*

This configuration is shown in Figure 8.1A. It is a simple exercise in similar triangles: just check that $OA \cdot OA^* = r^2$.

This is all fine and well, but it does not provide any clue why we should care about inversion. Inversion is not very interesting if we only look at one point at a time—how about two points A and B?

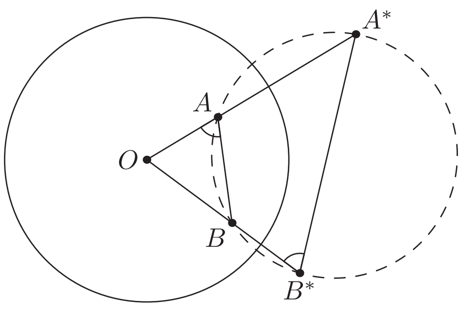

Figure 8.1B. Inversion preserves angles, kind of.

This situation is shown in Figure 8.1B. Now we have some more structure. Because $OA \cdot OA^* = OB \cdot OB^* = r^2$, by power of a point we see that quadrilateral ABB^*A^* is cyclic. Hence we obtain the following theorem.

Theorem 8.2 (Inversion and Angles). *If A^* and B^* are the inverses of A and B under inversion centered at O, then $\measuredangle OAB = -\measuredangle OB^*A^*$.*

Unfortunately, this does not generalize nicely* to arbitrary angles, as the theorem only handles angles with one vertex at O.

It is worth remarking how unimportant the particular value of r has been so far. Indeed, we see that often the radius is ignored altogether; in this case, we refer to this as **inversion**

* The correct generalization is to define an angle between two clines to be the angle formed by the tangents at an intersection point. This happens to be preserved under inversion. However, this is in general not as useful.

around P, meaning that we invert with respect to a circle centered at P with any positive radius. (After all, scaling r is equivalent to just applying a homothety with ratio r^2.)

Problem for this Section

Problem 8.3. If z is a nonzero complex number, show that the inverse of z with respect to the unit circle is $(\overline{z})^{-1}$.

8.2 Where Do Clines Go?

So far we have derived only a few very basic properties of inversion, nothing that would suggest it could be a viable method of attack for a problem. The results of this section will change that.

Rather than looking at just one or two points, we consider entire clines. The simplest example is a just a line through O.

Proposition 8.4. *A line passing through O inverts to itself.*

By this we mean that if we take each point on a line ℓ (including O and P_∞) and invert it, then look at the resulting locus of points, we get ℓ back again. The proof is clear.

What about a line not passing through O? Surprisingly, it is a circle! See Figure 8.2A

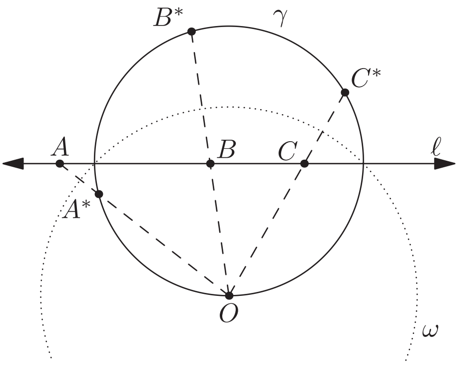

Figure 8.2A. A line inverts to a circle through O, and vice versa.

Proposition 8.5. *The inverse of a line ℓ not passing through O is a circle γ passing through O. Furthermore, the line through O perpendicular to ℓ passes through the center of γ.*

Proof. Let ℓ^* be the inverse of our line. Because P_∞ lies on ℓ, we must have O on ℓ^*. We show ℓ^* is a circle.

Let A, B, C be any three points on ℓ. It suffices to show that O, A^*, B^*, C^* are concyclic. This is easy enough. Because they are collinear, $\measuredangle OAB = \measuredangle OAC$. Using Theorem 8.2, $\measuredangle OB^*A^* = \measuredangle OC^*A^*$, as desired. Since any four points on ℓ^* are concyclic, that implies ℓ^* is just a circle.

It remains to show that ℓ is perpendicular to the line passing through the centers of ω (the circle we are inverting about) and γ. This is not hard to see in the picture. For a proof, let X be the point on ℓ closest to O (so $\overline{OX} \perp \ell$). Then X^* is the point on γ farthest from O, so that $\overline{OX^*}$ is a diameter of γ. Since O, X, X^* are collinear by definition, this implies the result. $\square$

In a completely analogous fashion one can derive the converse—the image of a circle passing through O is a line. Also, notice how the points on ω are fixed during the whole transformation.

This begs the question—what happens to the other circles? It turns out that these circles also invert to circles. Our proof here is of a different style than the previous one (although the previous proof can be rewritten to look more like this one). Refer to Figure 8.2B.

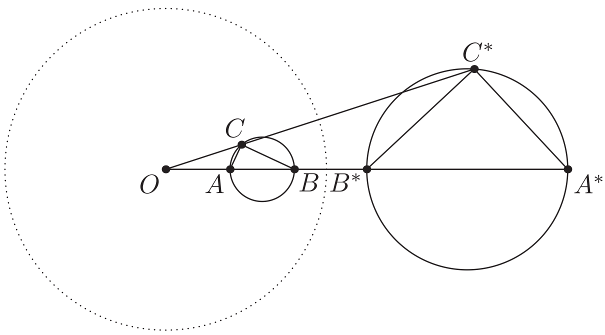

Figure 8.2B. A circle inverts to another circle.

Proposition 8.6. *Let γ be a circle not passing through O. Then γ^* is also a circle and does not contain O.*

Proof. Because neither O nor P_∞ is on γ, the inverse γ^* cannot contain these points either. Now, let $\overline{AB}$ be a diameter of γ with O on line AB (and $A, B \neq O$). It suffices to prove that γ^* is a circle with diameter $\overline{A^*B^*}$.

Consider any point C on γ. Observe that

$$90^\circ = \measuredangle BCA = -\measuredangle OCB + \measuredangle OCA.$$

By Theorem 8.2, we see that $-\measuredangle OCA = \measuredangle OA^*C^*$ and $-\measuredangle OCB = \measuredangle OB^*C^*$. Hence, a quick angle chase gives

$$90^\circ = \measuredangle OB^*C^* - \measuredangle OA^*C^* = \measuredangle A^*B^*C^* - \measuredangle B^*A^*C^* = -\measuredangle B^*C^*A^*$$

and hence C^* lies on the circle with diameter $\overline{A^*B^*}$. By similar work, any point on γ^* has inverse lying on γ, and we are done. $\square$

It is worth noting that the centers of these circles are also collinear. (However, keep in mind that the centers of the circle do not map to each other!)

We can summarize our findings in the following lemma.

Theorem 8.7 (Images of Clines). *A cline inverts to a cline. Specifically, in an inversion through a circle with center O,*

(a) A line through O inverts to itself.
(b) A circle through O inverts to a line (not through O), and vice versa. The diameter of this circle containing O is perpendicular to the line.
(c) A circle not through O inverts to another circle not through O. The centers of these circles are collinear with O.

We promised that inversion gives the power to turn circles into lines. This is a result of (b)—if we invert through a point with many circles, then all those circles become lines.

Finally, one important remark. Tangent clines (that is, clines which intersect exactly once, including at P_∞ in the case of two lines) remain tangent under inversion. This has the power to send tangent circles to parallel lines—we simply invert around the point at which they are internally or externally tangent.

Problems for this Section

Problem 8.8. In Figure 8.2C, sketch the inverse of the five solid clines (two lines and three circles) about the dotted circle ω. **Hint**: 279

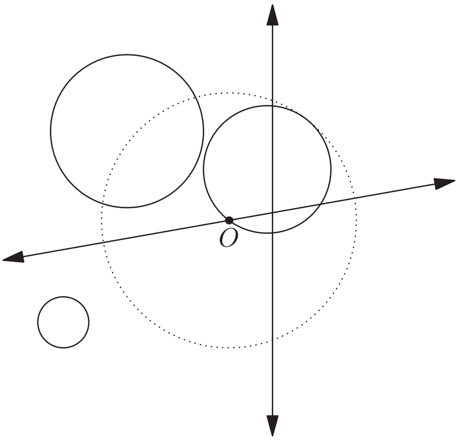

Figure 8.2C. Practice inverting.

Lemma 8.9 (Inverting an Orthocenter). *Let ABC be a triangle with orthocenter H and altitudes $\overline{AD}$, $\overline{BE}$, $\overline{CF}$. Perform an inversion around C with radius $\sqrt{CH \cdot CF}$. Where do the six points each go?* **Hint**: 257

Lemma 8.10 (Inverting a Circumcenter). *Let ABC be a triangle with circumcenter O. Invert around C with radius 1. What is the relation between O^*, C, A^*, and B^*?* **Hint**: 252

Lemma 8.11 (Inverting the Incircle). *Let ABC be a triangle with circumcircle Γ and contact triangle DEF. Consider an inversion with respect to the incircle of triangle ABC. Show that Γ is sent to the nine-point circle of triangle DEF.* **Hint**: 560

8.3 An Example from the USAMO

An example at this point would likely be illuminating. We revisit a problem first given in Chapter 3.

Example 8.12 (USAMO 1993/2). Let $ABCD$ be a quadrilateral whose diagonals $\overline{AC}$ and $\overline{BD}$ are perpendicular and intersect at E. Prove that the reflections of E across $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$ are concyclic.

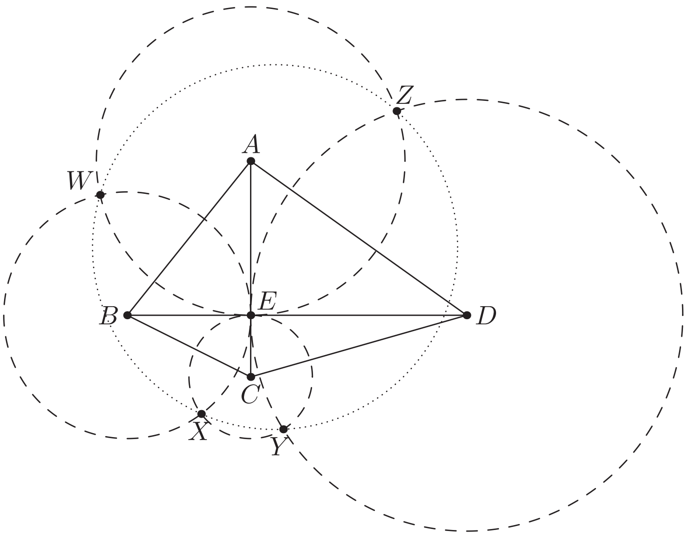

Figure 8.3A. Adding in some circles.

Let the reflections respectively be W, X, Y, Z.

At first, this problem seems a strange candidate for inversion. Indeed, there are no circles. Nevertheless, upon thinking about the reflection condition one might notice

$$AW = AE = AZ$$

which motivates us to construct a circle ω_A centered at A passing through all three points. If we define ω_B, ω_C, and ω_D similarly, suddenly we no longer have to worry about reflections. W is the just the second intersection of ω_A and ω_B, and so on.

Let us rephrase this problem in steps now.

1. Let $ABCD$ be a quadrilateral with perpendicular diagonals that meet at E.
2. Let ω_A be a circle centered at A through E.
3. Define ω_B, ω_C, ω_D similarly.
4. Let W be the intersection of ω_A and ω_B other than E.
5. Define X, Y, Z similarly.
6. Prove that $WXYZ$ is concyclic.

At this point, it may not be clear why we want to invert. Many students learning inversion for the first time are tempted to invert about ω_A. As far as I can tell, this leads nowhere, because it misses out on one of the most compelling reasons to invert:

Inversion lets us turn circles into lines.

This is why inversion around ω_A seems fruitless. There are few (read: zero) circles passing through A, so all the circles in the figure stay as circles, while some former lines become new circles. Hence inverting about ω_A is counterproductive: the resulting problem is more complicated than the original!

So what point has a lot of circles passing through it? Well, how about E? All four circles pass through it. Hence, we invert around a circle centered at E with radius 1. (Just because a point has no circle around it does not prevent us from using it as the center of inversion!)

What happens to each of the mapped points? Let us consider it step-by-step.

1. $A^*B^*C^*D^*$ is still some quadrilateral. As A^*, and C^* stay on line AC, and B^* and D^* stay on line BD, we have that $A^*B^*C^*D^*$ also has perpendicular diagonals meeting at E. Since $ABCD$ is arbitrary, we likewise treat $A^*B^*C^*D^*$ as arbitrary.[†]
2. ω_A passes through E, so it maps to some line perpendicular to line EA. This is not enough information to determine ω_A^* yet—what is the point of intersection ω_A^* has with line EA? Actually, it is the midpoint of $\overline{A^*E}$. For let M_A be the point diametrically opposite E on ω_A; this is the pre-image of the their intersection. Now A is the midpoint of $\overline{M_AE}$, so M_A^* is the midpoint of $\overline{A^*E}$.
   In other words, ω_A^* is the perpendicular bisector of $\overline{A^*E}$.
3. Define ω_B^*, ω_C^*, ω_D^* similarly.
4. W^* is the intersection of the two lines ω_A^* and ω_B^*, simply because W is the intersection of ω_A and ω_B other than E. (Of course, ω_A^* and ω_B^* also meet at the point at infinity, which is the image of E.)
5. X^*, Y^*, Z^* are also defined similarly.
6. We wish to show $WXYZ$ is cyclic. By Theorem 8.7, this is equivalent to showing $W^*X^*Y^*Z^*$ is cyclic.

This is the thought process for inverting a problem. We consider the steps used to construct the original problem, and one by one find their inversive analogs. While perhaps not easy at first, this requires no ingenuity and is a skill that can be picked up with enough practice, since it is really just a mechanical calculation.

Figure 8.3B shows the completed diagram.

We are just moments from finishing. We wish to show that quadrilateral $W^*X^*Y^*Z^*$ is cyclic. But it is a rectangle, so this is obvious!

Solution to Example 8.12. Define ω_A, ω_B, ω_C, ω_D to be circles centered at A, B, C, D passing through E. Observe that W is the second intersection of ω_A and ω_B, et cetera.

[†] Degrees of freedom, anyone? When you are considering the inverted version of a problem, you want to make sure the number of degrees of freedom does not change. See Section 5.3 for more discussion on degrees of freedom.

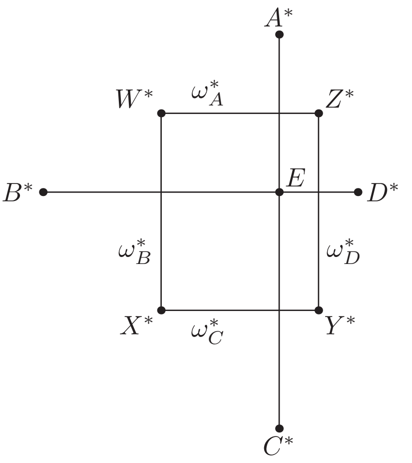

Figure 8.3B. Inverting the USAMO.

Consider an inversion at E. It maps $\omega_A, \omega_B, \omega_C, \omega_D$ to four lines which are the sides of a rectangle. Hence the images of W, X, Y, Z under this inversion form a rectangle, which in particular is cyclic. Inverting back, $WXYZ$ is cyclic as desired. $\square$

Notice that we do not have to go through the full detail in explaining how to arrive at the inverted image. In a contest, it is usually permissible to just state the inverted problem, since deriving the inverted figure is a straightforward process.

Usually an inverted problem will not be *this* easy.[‡] However, we often have good reason to believe that the inverted problem is simpler than the original. In the above example, the opportunity to get rid of all the circles motivated our inversion at E, and indeed we found the resulting problem to be trivial.

8.4 Overlays and Orthogonal Circles

Consider two circles ω_1 and ω_2 with centers O_1 and O_2 intersecting at two points X and Y. We say they are **orthogonal** if

$$\angle O_1 X O_2 = 90^\circ,$$

i.e., the lines O_1X and O_1Y are the tangents to the second circle. Of course, ω_1 is orthogonal to ω_2 if and only if ω_2 is orthogonal to ω_1.

It is clear that if ω_2 is a circle and O_1 a point outside it, we can draw a unique circle centered at O_1 orthogonal to ω_2: namely, the circle whose radius is equal to the length of the tangent to ω_2.

Orthogonal circles are nice because of the following lemma.

[‡] But you can certainly find other examples. At the 2014 IMO, one of my teammates said that he was looking for problems that were trivialized by inversion. Another friend responded that this was easy—just take a trivial problem and invert it!

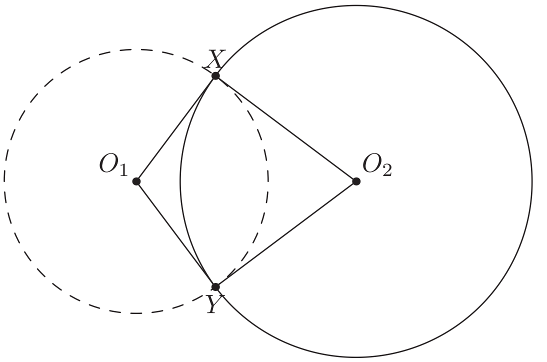

Figure 8.4A. Two orthogonal circles.

Lemma 8.13 (Inverting Orthogonal Circles). *Let ω and γ be orthogonal circles. Then γ inverts to itself under inversion with respect to ω.*

Proof. This is a consequence of power of a point. Let ω and γ intersect at X and Y, and denote by O the center ω. Consider a line through O intersecting γ at A and B. Then

$$OX^2 = OA \cdot OB$$

but since OX is the radius ω, A inverts to B. $\square$

What's the upshot? When a figure inverts to itself, we get to exploit what I call the "inversion overlay principle". Loosely, it goes as follows:

Problems that invert to themselves are usually really easy.

There are a few ways this can happen. Sometimes it is because we force a certain circle to be orthogonal. Other times it is a good choice of radius that plays well with the problem. In either case the point is that we gain information by overlaying the inverted diagram onto the original.

Here is the most classical example of overlaying, called a **Pappus chain** embedded in a **shoemaker's knife**. See Figure 8.4B.

Example 8.14 (Shoemaker's Knife). Let A, B, C be three collinear points (in that order) and construct three semicircles Γ_{AC}, Γ_{AB}, ω_0, on the same side of $\overline{AC}$, with diameters $\overline{AC}$, $\overline{AB}$, $\overline{BC}$, respectively. For each positive integer k, let ω_k be the circle tangent to Γ_{AC} and Γ_{AB} as well as ω_{k-1}.

Let n be a positive integer. Prove that the distance from the center of ω_n to $\overline{AC}$ is n times its diameter.

The point of inverting is to handle the abominable tangency conditions. Note that each ω_i is tangent to both Γ_{AB} and Γ_{AC}, so it makes sense to force both of these circles into lines. This suggests inverting about A. As an added bonus, these two lines become parallel.

It is perhaps not clear yet what to use as the radius, or even if we need to pick a radius. However, we want to ensure that the diameter of ω_n remains a meaningful quantity after the inversion. This suggests keeping ω_n fixed.

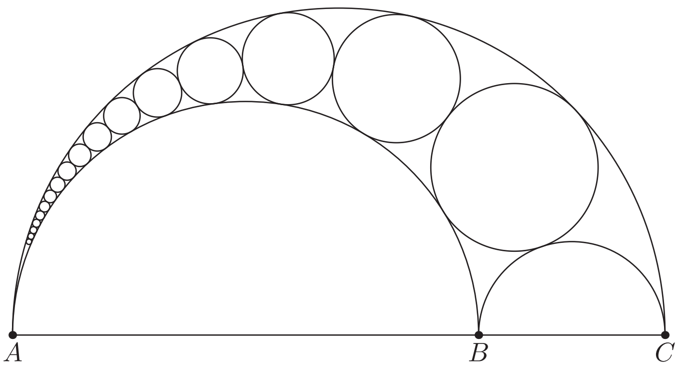

Figure 8.4B. The Shoemaker's Knife.

This motivates us to invert around A with radius r in such a way that ω_n is orthogonal to our circle of inversion. What effect does this have?

- ω_n stays put, by construction.
- The semicircles Γ_{AB} and Γ_{AC} pass through A, so their images Γ_{AB}^* and Γ_{AC}^* are lines perpendicular to line AC.
- All the other ω_i are now circles tangent to these two lines.

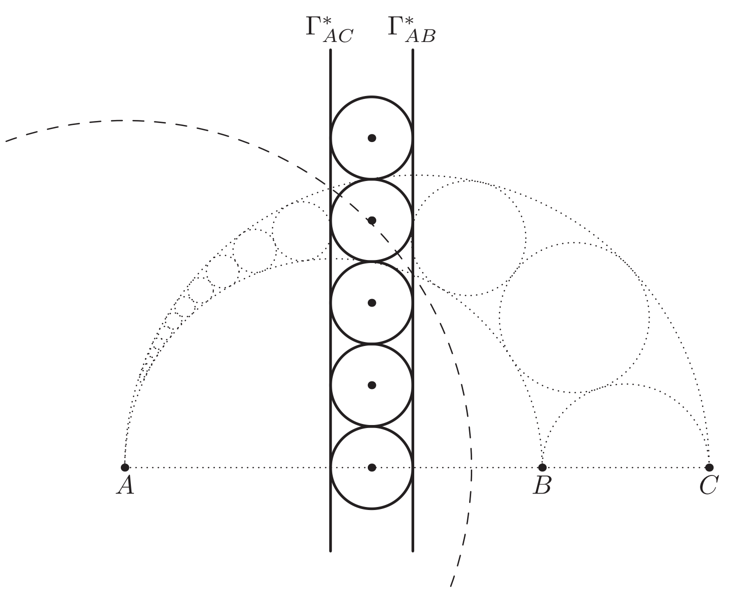

Figure 8.4C. Inverting with ω_3 fixed (so $n = 3$). We invert around the dashed circle centered at A, orthogonal to ω_3.

Figure 8.4C shows the inverted image, overlaid on the original image. The two semicircles have become convenient parallel lines, and the circles of the Pappus chain line up obediently between them. Because the circles are all congruent, the conclusion is now obvious.

8.5 More Overlays

An example of the second type of overlay is the short inversive proof of Lemma 4.33 we promised.

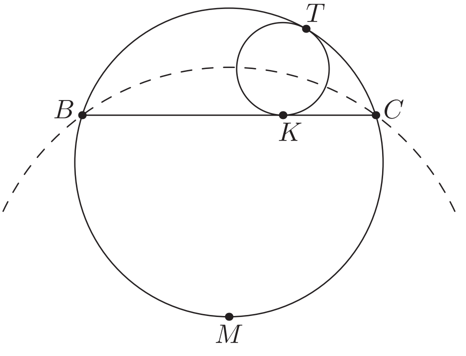

Figure 8.5A. Revisiting Lemma 4.33.

Example 8.15. Let $\overline{BC}$ be a chord of a circle Ω. Let ω be a circle tangent to chord $\overline{BC}$ at T and internally tangent to ω at T. Then ray TK passes through the midpoint M of the arc $\widehat{BC}$ not containing T. Moreover, MC^2 is the power of M with respect to ω.

Proof. Let Γ be the circle centered at M passing through B and C. What happens when we invert around Γ?

Firstly, Ω is a circle through M, so it gets sent to a line. Because B and C lie on Γ and are fixed by this inversion, it must be precisely the line BC. In particular, this implies line BC gets sent to Ω. In other words, the inversion simply swaps line BC and Γ.

Perhaps the ending is already obvious. We claim that ω just gets sent to itself. Because $\overline{BC}$ and Ω trade places, ω^* is also a circle tangent to both. Also, the centers of ω^* and ω are collinear with M. This is enough to force $\omega = \omega^*$. (Why?)

Now K is the tangency point of ω with $\overline{BC}$, so K^* is the tangency point of $\omega^* = \omega$ with $(MB^*C^*) = \Omega$. But this is T; hence K and T are inverses.

In particular, M, K, T are collinear and $MK \cdot MT = MC^2$. □

Here is a nice general trick that can force overlays when dealing with a triangle ABC.

Lemma 8.16 (Force-Overlaid Inversion). *Let ABC be a triangle. Consider the transformation consisting of an inversion about A with radius $\sqrt{AB \cdot AC}$, followed by a reflection around the angle bisector of $\angle BAC$. This transformation fixes B and C.*

The above demonstration applies the lemma with $A = M$. Because $\triangle BMC$ was isosceles, there was no need to use the additional reflection.

Fixing a triangle ABC is often very powerful since problems often build themselves around ABC. In particular, tangency to (ABC) is involved (as it becomes tangency to line BC). This led to the solution in the above example.

Problem for this Section

Problem 8.17. Work out the details in the proof of Lemma 8.16.

8.6 The Inversion Distance Formula

The inversion distance formula gives us a way to handle lengths in inversion. It is completely multiplicative, making it nice for use with ratios but more painful if addition is necessary.

Theorem 8.18 (Inversion Distance Formula). *Let A and B be points other than O and consider an inversion about O with radius r. Then*

$$A^*B^* = \frac{r^2}{OA \cdot OB} \cdot AB.$$

Equivalently,

$$AB = \frac{r^2}{OA^* \cdot OB^*} A^*B^*.$$

This first relation follows from the similar triangles we used in Figure 8.1B, and is left as an exercise. The second is a direct consequence of the first (why?).

The inversion distance formula is useful when you need to deal with a bunch of lengths. See Problem 8.20.

Problems for this Section

Problem 8.19. Prove the inversion distance formula.

Problem 8.20 (Ptolemy's Inequality)**.** For any four distinct points A, B, C, and D in a plane, no three collinear, prove that

$$AB \cdot CD + BC \cdot DA \geq AC \cdot BD.$$

Moreover, show that equality holds if and only if A, B, C, D lie on a circle in that order.
Hints: 118 136 539 130

8.7 More Example Problems

The first problem is taken from the Chinese Western Mathematical Olympiad.

Example 8.21 (Chinese Olympiad 2006). Let $ADBE$ be a quadrilateral inscribed in a circle with diameter $\overline{AB}$ whose diagonals meet at C. Let γ be the circumcircle of $\triangle BOD$, where O is the midpoint of $\overline{AB}$. Let F be on γ such that $\overline{OF}$ is a diameter of γ, and let ray FC meet γ again at G. Prove that A, O, G, E are concyclic.

We are motivated to consider inversion by the two circles passing through O, as well as the fact that O itself is a center of a circle through many points. Inversion through O also preserves the diameter $\overline{AB}$, which is of course important.

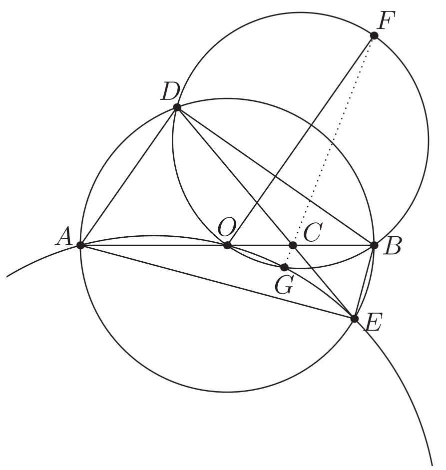

Figure 8.7A. Show that $OAEG$ is concyclic.

Before inverting, though, let us rewrite the problem with phantom point G_1 as the intersection of (OFB) and (OAE), and attempt to prove instead that F, C, G_1 are collinear. This lets us define G_1^* as the intersection of two lines.

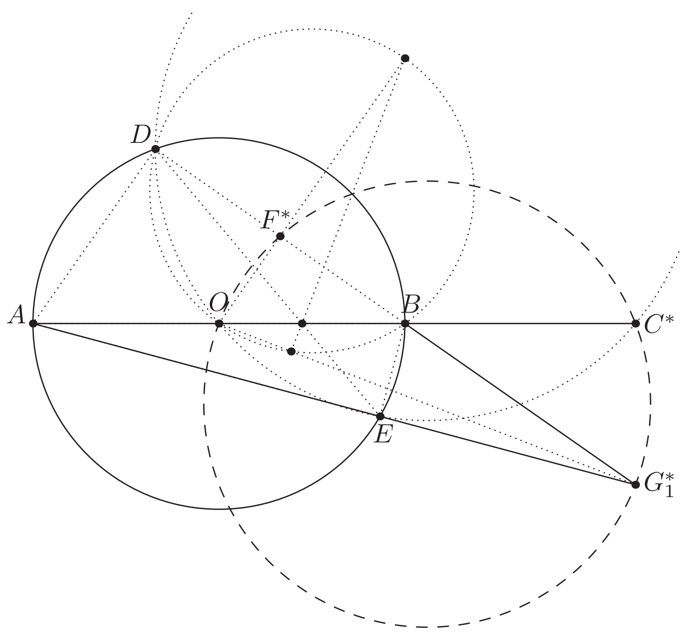

Figure 8.7B. In the inverted image, we wish to show that points O, F^*, C^*, G_1^* are cyclic.

We now invert around the circle with diameter $\overline{AB}$. We figure out where each point goes.

1. Points D, B, A, E stay put, because they lie on the circle we are inverting around. So $D^* = D$, etc.

2. C was the intersection of $\overline{AB}$ and $\overline{DE}$. Hence C^* is a point on line AB so that C^*DOE is cyclic.
3. F is the point diametrically opposite O on (BOD). That means that $\angle ODF = 90^\circ$. So, $\angle OF^*D^* = 90^\circ$. Similarly, $\angle OF^*B^* = 90^\circ$. Hence, F^* is just the midpoint of $\overline{DB}$!
4. G_1 is defined as the intersection of (OFB) and (OAE), so G_1^* is the intersection of lines F^*B and AE.
5. We wish to show that O, F^*, C^*, and G_1 are concyclic.

Okay. Well, $\overline{OF^*} \perp \overline{BD}$; thus to prove O, F^*, C^*, G_1^* are concyclic, it suffices to show that $\overline{G_1^*C^*} \perp \overline{AC^*}$. Now look once more at circle $(OEDC^*)$. Notice something?

Because $\overline{AD} \perp \overline{BG_1^*}$, $\overline{BE} \perp \overline{AG_1^*}$, and O is the midpoint of $\overline{AB}$, we discover this is the nine-point circle of $\triangle ABG_1^*$. We are done.

Solution to Example 8.21. Let G_1 be the intersection of (ODB) and (OAE) and invert around the circle with diameter $\overline{AB}$. In the inverted image, F^* is the midpoint of $\overline{BD}$, C^* lies on line AB and (DOE), and G^* is the intersection of lines DB and AE. We wish to show O, F^*, C^*, G_1^* are cyclic.

Because (OED) is the nine-point circle of $\triangle ABG_1^*$, we see C^* is the foot of G_1^* onto line AB. On the other hand, $\angle OF^*B = 90^\circ$ as well so we are done. □

Let us conclude by examining the fifth problem from the 2009 USA olympiad.

Example 8.22 (USAMO 2009/5). Trapezoid $ABCD$, with $\overline{AB} \parallel \overline{CD}$, is inscribed in circle ω and point G lies inside triangle BCD. Rays AG and BG meet ω again at points P and Q, respectively. Let the line through G parallel to $\overline{AB}$ intersect $\overline{BD}$ and $\overline{BC}$ at points R and S, respectively. Prove that quadrilateral $PQRS$ is cyclic if and only if $\overline{BG}$ bisects $\angle CBD$.

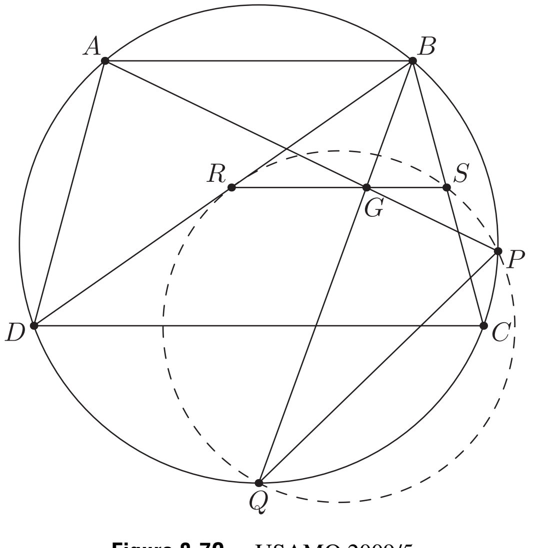

Figure 8.7C. USAMO 2009/5.

The main reason we might want to attempt inversion is that there are not just four, or even five, but six points all lying on one circle. It would be great if we could make that circle into a line.

So if we are going to invert, we should do so around a point on the circle ω. Because we have a bisector at $\angle CBD$, it makes sense to invert around B in order to keep this condition nice. Also, the parallel lines become tangent circles at B. More plainly, there are just a lot of lines passing through B.

Again we work out what happens in steps.

1. Cyclic quadrilateral $ABCD$ becomes a point B and three points A^*, C^*, D^* on a line in that order. Because $\overline{AB} \parallel \overline{CD}$, we actually see that $\overline{A^*B}$ is tangent to (BC^*D^*).
2. G is an arbitrary point inside triangle BCD. That means G^* is some point inside $\angle C^*BD^*$, but outside triangle BC^*D^*.
3. R and S are the intersections of a parallel line through G with $\overline{BD}$ and $\overline{BC}$. Therefore R^* is the intersection of a circle tangent to (BC^*D^*) at B (this is the image of parallel lines) with ray BD^*. S^* is the intersection of this same circle with ray BS^*.
4. Q was the intersection of $(ABCD)$ with ray BG, so now Q^* is the intersection of $\overline{BG^*}$ with the line through A^*, C^*, and D^*.
5. P was the intersection of $(ABCD)$ with line AG. Hence P^* is the point on line A^*C^* such that $BA^*G^*P^*$ is cyclic.
6. We wish to show that $P^*Q^*R^*S^*$ is cyclic if and only if $\overline{BG^*}$ bisects $\angle R^*BS^*$.

The inverted diagram is shown in Figure 8.7D.

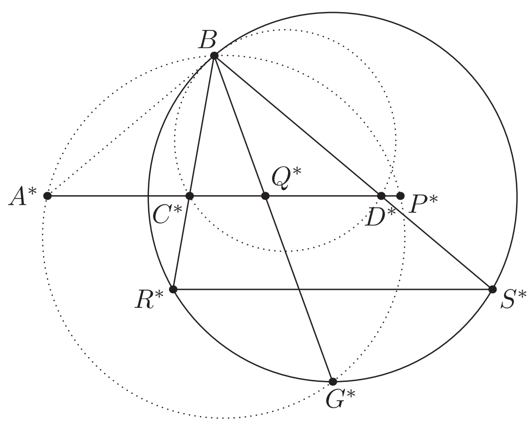

Figure 8.7D. Inverting the USAMO... again!

Now it appears that $\overline{P^*Q^*}$ is parallel to $\overline{S^*R^*}$. Actually, this is obvious, because there is a homothety at B taking $\overline{C^*D^*}$ to $\overline{S^*R^*}$. This is good for us, because now $P^*Q^*R^*S^*$ is cyclic if and only if it is isosceles.

We can also basically ignore (BC^*D^*) now; it is just there to give us these parallel lines. For that matter, we can more or less ignore C^* and D^* now too.

Let us eliminate the point A^*. We have

$$\measuredangle Q^*P^*G^* = \measuredangle A^*P^*G^* = \measuredangle A^*BG^* = \measuredangle BS^*G^*.$$

Seeing this, we extend line G^*P^* to meet (BS^*R^*) at X, as in Figure 8.7E. This way,

$$\measuredangle Q^*P^*G^* = \measuredangle BS^*G^* = \measuredangle BX^*G^*.$$

Therefore, $\overline{P^*Q^*} \parallel \overline{BX}$ holds unconditionally. This lets us get rid of P^* in the sense that it is just a simple intersection of $\overline{G^*X}$ and the parallel line; we can anchor the problem around (BXR^*S^*).

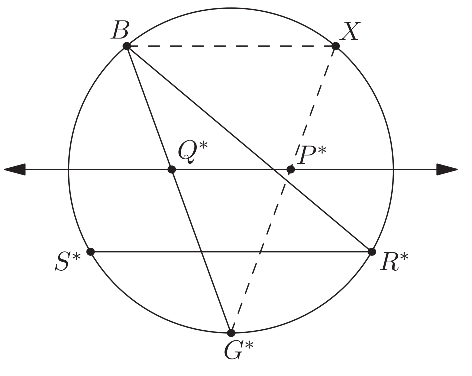

Figure 8.7E. Cleaning up the inverted diagram.

Thus, we have reduced the problem to the following.

Let BXS^*R^* be an isosceles trapezoid and ℓ a fixed line parallel to its bases. Let G^* be a point on its circumcircle and denote the intersections of ℓ with $\overline{BG^*}$ and $\overline{XG^*}$ by Q^* and P^*. Prove that $P^*S^* = Q^*R^*$ if and only if G^* is the midpoint of arc R^*S^*.

This is actually straightforward symmetry. See the solution below.

Solution to Example 8.22. Perform an inversion around B with arbitrary radius, and denote the inverse of a point Z with Z^*.

After inversion, we obtain a cyclic quadrilateral $BS^*G^*R^*$ and points C^*, D^* on $\overline{BS^*}$, $\overline{BR^*}$, such that (BC^*D^*) is tangent to $(BS^*G^*R^*)$—in other words, so that $\overline{C^*D^*}$ is parallel to $\overline{S^*R^*}$. Point A^* lies on line $\overline{C^*D^*}$ so that $\overline{A^*B}$ is tangent to $(BS^*G^*R^*)$. Points P^* and Q^* are the intersections of (A^*BG^*) and $\overline{BG^*}$ with line C^*D^*.

Observe that $P^*Q^*R^*S^*$ is a trapezoid, so it is cyclic if and only if it isosceles.

Let X be the second intersection of line G^*P^* with (BS^*R^*). Because $\measuredangle Q^*P^*G^* = \measuredangle A^*BG^* = \measuredangle BXG^*$, we find that BXS^*R^* is an isosceles trapezoid.

If G^* is indeed the midpoint of the arc then everything is clear by symmetry now. Conversely, if $P^*R^* = Q^*S^*$ then that means $P^*Q^*R^*S^*$ is a cyclic trapezoid, and hence that the perpendicular bisectors of $\overline{P^*Q^*}$ and $\overline{R^*S^*}$ are the same. Hence B, X, P^*, Q^* are symmetric around this line. This forces G^* to be the midpoint of arc R^*S^* as desired. □

These two examples demonstrate inversion as a means of transforming one problem into another (as opposed to some of the overlaying examples, which used both at once). It is almost like you are given a choice—which of these two problems looks easier, the inverted one or the original one? Which would you like to solve?

8.8 When to Invert

As a reminder, here are things inversion with a center O handles well. Hopefully these were clear from the examples.

- Clines tangent to each other. In particular, we can take a tangent pair of circles to two parallel lines.
- Several circles pass through O. Inverting around O eliminates the circles.
- Diagrams that invert to themselves! Overlaying an inverted diagram is frequently fruitful.

Here are things that inversion does not handle well.

- Scattered angles. Theorem 8.2 gives us control over angles that have a ray passing through a center O, but we do not have much control over general angles.
- Problems that mostly involve lines and not circles.

Finally, here is a reminder of what inversion through a circle ω with center O preserves (and what it does not).

- Points on ω are fixed.
- Clines are sent to clines. Moreover,
  - If a circle γ is mapped to a line ℓ, then ℓ is perpendicular to the line joining O to the center of γ.
  - If a circle γ is mapped to γ^*, the center of γ is not in general the center of γ^*. It is true, however, that the centers of γ and γ^* are collinear with the center of inversion.
- Tangency and intersections are preserved.

8.9 Problems

Problem 8.23. Let ABC be a right triangle with $\angle C = 90^\circ$ and let X and Y be points in the interiors of $\overline{CA}$ and $\overline{CB}$, respectively. Construct four circles passing through C, centered at A, B, X, Y. Prove that the four points lying on at exactly two of these four circles are concyclic. (See Figure 8.9A.) **Hints**: 198 626 178 577

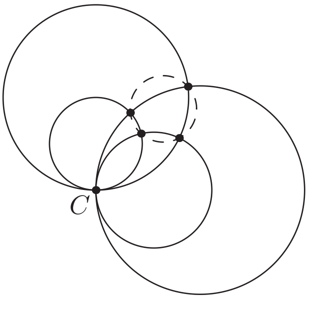

Figure 8.9A. The four intersections are concyclic (dashed circle).

Problem 8.24. Let $\omega_1, \omega_2, \omega_3, \omega_4$ be circles with consecutive pairs tangent at A, B, C, D, as shown in Figure 8.9B. Prove that quadrilateral $ABCD$ is cyclic. **Hints**: 294 677 172 **Sol:** p.272

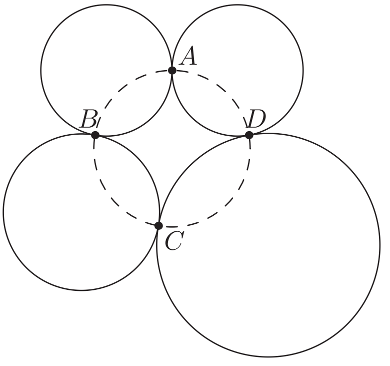

Figure 8.9B. Is there a connection between this and Theorem 2.25?

Problem 8.25. Let A, B, C be three collinear points and P be a point not on this line. Prove that the circumcenters of $\triangle PAB$, $\triangle PBC$, and $\triangle PCA$ lie on a circle passing through P. **Hints**: 465 536 496

Problem 8.26 (BAMO 2008/6). A point D lies inside triangle ABC. Let A_1, B_1, C_1 be the second intersection points of the lines AD, BD, and CD with the circumcircles of BDC, CDA, and ADB, respectively. Prove that

$$\frac{AD}{AA_1} + \frac{BD}{BB_1} + \frac{CD}{CC_1} = 1.$$

Hints: 439 170 256

Problem 8.27 (Iran Olympiad 1996). Consider a semicircle with center O and diameter $\overline{AB}$. A line intersects line AB at M and the semicircle at C and D such that $MC > MD$ and $MB < MA$. Suppose (AOC) and (BOD) meet at a point K other than O. Prove that $\angle MKO = 90^\circ$. **Hints**: 403 27 **Sol:** p.272

Problem 8.28 (Shortlist 2003/G4). Let $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ be distinct circles such that Γ_1, Γ_3 are externally tangent at P, and Γ_2, Γ_4 are externally tangent at the same point P. Suppose that Γ_1 and Γ_2, Γ_2 and Γ_3, Γ_3 and Γ_4, Γ_4 and Γ_1 meet at A, B, C, D, respectively, and that all these points are different from P. Prove that

$$\frac{AB \cdot BC}{AD \cdot DC} = \frac{PB^2}{PD^2}.$$

Hints: 120 247 22

Problem 8.29. Let ABC be a triangle with incenter I and circumcenter O. Prove that line IO passes through the centroid G_1 of the contact triangle. **Hints**: 532 323 579

Problem 8.30 (NIMO 2014). Let ABC be a triangle and let Q be a point such that $\overline{AB} \perp \overline{QB}$ and $\overline{AC} \perp \overline{QC}$. A circle with center I is inscribed in $\triangle ABC$, and is tangent to

$\overline{BC}$, $\overline{CA}$, and $\overline{AB}$ at points D, E, and F, respectively. If ray QI intersects $\overline{EF}$ at P, prove that $\overline{DP} \perp \overline{EF}$. **Hints**: 362 125 578 663 **Sol:** p.273

Problem 8.31 (EGMO 2013/5). Let Ω be the circumcircle of the triangle ABC. The circle ω is tangent to the sides AC and BC, and it is internally tangent to the circle Ω at the point P. A line parallel to AB intersecting the interior of triangle ABC is tangent to ω at Q. Prove that $\angle ACP = \angle QCB$. **Hints**: 282 449 255 143 **Sol:** p.273

Problem 8.32 (Russian Olympiad 2009). In triangle ABC with circumcircle Ω, the internal angle bisector of $\angle A$ intersects $\overline{BC}$ at D and Ω again at E. The circle with diameter $\overline{DE}$ meets Ω again at F. Prove that $\overline{AF}$ is a symmedian of triangle ABC. **Hints**: 594 648 321

Problem 8.33 (Shortlist 1997). Let $A_1A_2A_3$ be a non-isosceles triangle with incenter I. Let C_i, $i = 1, 2, 3$, be the smaller circle through I tangent to A_iA_{i+1} and A_iA_{i+2} (indices taken mod 3). Let B_i, $i = 1, 2, 3$, be the second point of intersection of C_{i+1} and C_{i+2}. Prove that the circumcenters of the triangles A_1B_1I, A_2B_2I, A_3B_3I are collinear. **Hints**: 76 242 620 561

Problem 8.34 (IMO 1993/2). Let A, B, C, D be four points in the plane, with C and D on the same side of the line AB, such that $AC \cdot BD = AD \cdot BC$ and $\angle ADB = 90^\circ + \angle ACB$. Find the ratio $\frac{AB \cdot CD}{AC \cdot BD}$, and prove that the circumcircles of the triangles ACD and BCD are orthogonal. **Hints**: 7 384 322 3

Problem 8.35 (IMO 1996/2). Let P be a point inside a triangle ABC such that

$$\angle APB - \angle ACB = \angle APC - \angle ABC.$$

Let D, E be the incenters of triangles APB, APC, respectively. Show that the lines AP, BD, CE concur. **Hints**: 581 638 338 341

Problem 8.36 (IMO 2015/3). Let ABC be an acute triangle with $AB > AC$. Let Γ be its cirumcircle, H its orthocenter, and F the foot of the altitude from A. Let M be the midpoint of $\overline{BC}$. Let Q be the point on Γ such that $\angle HQA = 90^\circ$ and let K be the point on Γ such that $\angle HKQ = 90^\circ$. Assume that the points A, B, C, K, and Q are all different and lie on Γ in this order. Prove that the circumcircles of triangles KQH and FKM are tangent to each other. **Hints**: 402 673 324 400 155 **Sol:** p.274

Problem 8.37 (ELMO Shortlist 2013). Let ω_1 and ω_2 be two orthogonal circles, and let the center of ω_1 be O. Diameter $\overline{AB}$ of ω_1 is selected so that B lies strictly inside ω_2. The two circles tangent to ω_2 through both O and A touch ω_2 at F and G. Prove that quadrilateral $FOGB$ is cyclic. **Hints**: 96 353 112 **Sol:** p.274

CHAPTER 9

Projective Geometry

Projective geometry is all geometry. — Arthur Cayley

In the previous chapter we studied inversion, a transformation that deals with circles. It also happened to nicely preserve incidence, i.e., inversion preserves intersections. Projective geometry features a powerful set of tools that this time focus primarily on analyzing incidence. Problems that mostly deal with intersections, parallel lines, tangent circles, and so on, often succumb to projective geometry.

9.1 Completing the Plane

First, we set up the projective plane with points at infinity.

Imagine we are walking down the infinitely long corridor in Figure 9.1A and take a moment to look around us.

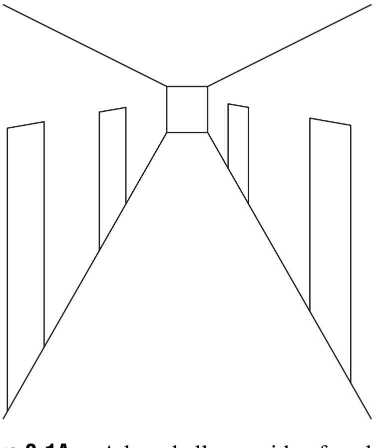

Figure 9.1A. A long hallway with a few doors.

There are some parallel lines in the figure, say the two lines that mark the floor. But they are not actually parallel in the picture: the two lines are converging towards a point. In fact, all the parallel lines are converging towards the same point on the horizon. So it does seem like parallel lines intersect infinitely far away, even in a plane (for example, consider the left wall or the right wall).

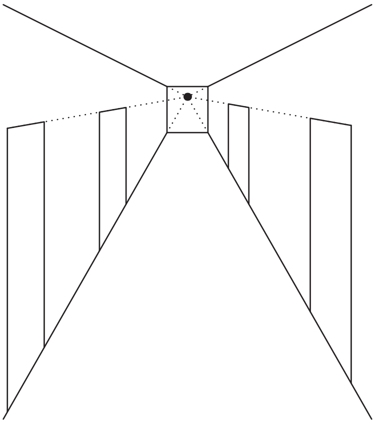

Figure 9.1B. Are the parallel lines really parallel?

The **real projective plane** uses precisely this idea. In addition to the standard points of Euclidean plane (which we call **Euclidean points**), it also includes a **point at infinity** for each class of parallel lines (one can think of this as adding a point at infinity for each direction). To be more precise, we partition all the lines of the Euclidean plane into equivalence classes (called **pencils** of parallel lines) where two distinct lines are in the same class if they are parallel. Then we add a point at infinity for each pencil. We also add one extra line, the **line at infinity**, comprising exactly of all the points at infinity.

With this modification, any two lines do in fact intersect at exactly one point. The intersection of two non-parallel lines is a Euclidean point, while two parallel lines meet at the point at infinity. The use of this convention lets us replace the clumsy language of "concurrent or all parallel" (as in Theorem 2.9).

Finally, throughout this chapter we use a special shorthand. For points A, B, C, D, let $\overline{AB} \cap \overline{CD}$ denote the intersection of lines AB and CD, possibly at infinity.

9.2 Cross Ratios

The **cross ratio** is an important invariant in projective geometry. Given four collinear points A, B, X, Y (which may be points at infinity), we define the cross ratio as

$$(A, B; X, Y) = \frac{XA}{XB} \div \frac{YA}{YB}.$$

Here the ratios are directed with the same convention as Menelaus's theorem; in particular, the cross ratio can be negative! If A, B, X, Y lie on a number line then this can be written as

$$(A, B; X, Y) = \frac{x - a}{x - b} \div \frac{y - a}{y - b}.$$

You can check that $(A, B; X, Y) > 0$ precisely when segments $\overline{AB}$ and $\overline{XY}$ are disjoint or one is contained inside the other. We also generally assume $A \neq X$, $B \neq X$, $A \neq Y$, $B \neq Y$.

We can also define the cross ratio for four lines a, b, x, y concurrent at some point P. If $\angle(\ell, m)$ is the angle between the two lines ℓ and m, then we can write

$$(a, b; x, y) = \pm \frac{\sin \angle(x, a)}{\sin \angle(x, b)} \div \frac{\sin \angle(y, a)}{\sin \angle(y, b)}.$$

The sign is chosen in a similar manner as the procedure for four points: if one of the four angles formed by line a and b contains neither x nor y, then $(a, b; x, y)$ is positive; otherwise it is negative.

If A, B, X, Y are collinear points on lines a, b, x, y (respectively) concurrent at P, we write

$$P(A, B; X, Y) = (a, b; x, y).$$

The structure $P(A, B; X, Y)$ is called a **pencil** of lines. See Figure 9.2A.

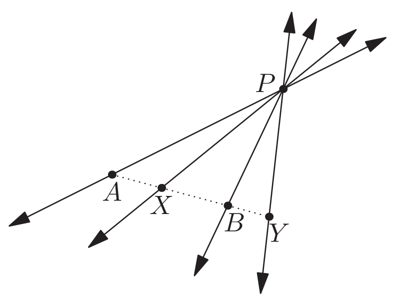

Figure 9.2A. Actually, $P(A, B; X, Y) = (A, B; X, Y)$.

As you might have already guessed, the sign convention for the trigonometric form is just contrived so that the following theorem holds.

Theorem 9.1 (Cross-Ratio Under Perspectivity). *Suppose that $P(A, B; X, Y)$ is a pencil of lines. If A, B, X, Y are collinear then*

$$P(A, B; X, Y) = (A, B; X, Y).$$

Proof. This is just a computation with the law of sines on $\triangle XPA$, $\triangle XPB$, $\triangle YPA$, $\triangle YPB$. There are multiple configurations to check, but they are not so different. $\square$

We can even define the cross ratio for four points on a circle, as follows:

Theorem 9.2 (Cross Ratios on Cyclic Quadrilaterals). *Let A, B, X, Y be concyclic. If P is any point on its circumcircle, then $P(A, B; X, Y)$ does not depend on P. Moreover,*

$$P(A, B; X, Y) = \pm \frac{XA}{XB} \div \frac{YA}{YB}$$

where the sign is positive if $\overline{AB}$ and $\overline{XY}$ do not intersect, and negative otherwise.

The invariance just follows from the fact that the angles are preserved as P varies around the circle. Hence, we just define the cross ratio of four concyclic points to be the value of $P(A, B; X, Y)$ for any particular P. The actual ratio $\frac{XA}{XB} : \frac{YA}{YB}$ follows by applying the law of sines and the details are left as an exercise.

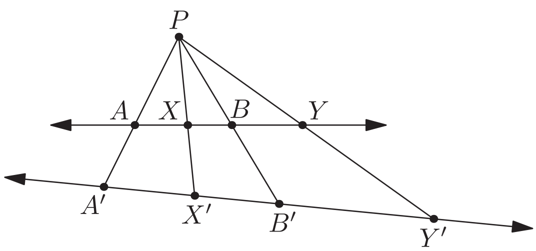

Figure 9.2B. Taking perspectivity at P.

Why do we care? Consider the situation in Figure 9.2B. Two lines ℓ and m are given, and points A, B, X, Y are on ℓ. We can pick any point P and consider the intersections of lines PA, PB, PX, PY with m, say A', B', X', Y'. Then

$$(A, B; X, Y) = P(A, B; X, Y) = P(A', B'; X', Y') = (A', B'; X', Y').$$

In effect, that means we have the power to project $(A, B; X, Y)$ from line ℓ onto line m. This is called taking **perspectivity** at P. We often denote this by

$$(A, B; X, Y) \stackrel{P}{=} (A', B'; X', Y').$$

The same technique can be done if P, A, X, B, Y are concyclic, in which case we may project onto a line. Conversely, given $(A, B; X, Y)$ on a line we may pull from P onto circle through P, as in Figure 9.2C (and vice versa). The important thing is that these operations all preserve the cross ratio $(A, B; X, Y)$.

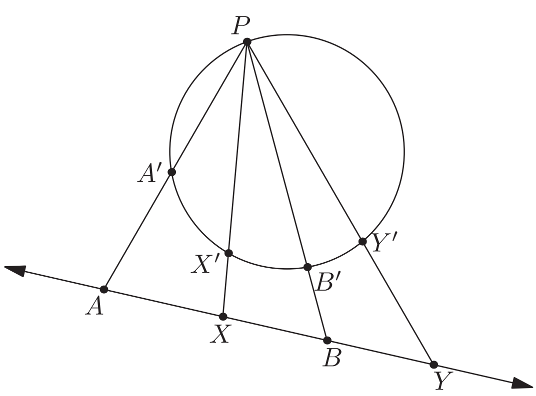

Figure 9.2C. Projecting via P from a line onto a circle through P.

The fact that cross ratio is preserved under all of these is why it is well-suited for problems that deal with lots of intersections. One can even think of chasing cross ratios around the diagram by repeatedly applying perspectives. We see more of this in later examples.

In the next section we investigate the most important case of the cross ratio, the harmonic bundle.

Problems for this Section

Problem 9.3. Check that

$$(A, B; X, Y) = (B, A; X, Y)^{-1} = (A, B; Y, X)^{-1} = (X, Y; A, B).$$

for any four distinct points A, B, X, Y.

Problem 9.4. Let A, B, X be distinct collinear points and k a real number. Prove that there is exactly one point Y (possibly the point at infinity) such that $(A, B; X, Y) = k$. **Hint**: 287

Problem 9.5. In Figure 9.2A, is $P(A, B; X, Y)$ positive or negative? **Hint**: 83

Problem 9.6. Let A, B, X be collinear points and P_∞ a point at infinity along their common line. What is $(A, B; X, P_\infty)$? **Hint**: 666

Problem 9.7. Give the proof of Theorem 9.2.

9.3 Harmonic Bundles

The most important case of our cross ratio is when $(A, B; X, Y) = -1$. We say that $(A, B; X, Y)$ is a **harmonic bundle** in this case, or just **harmonic**. Furthermore, a cyclic quadrilateral $AXBY$ is a **harmonic quadrilateral** if $(A, B; X, Y) = -1$.

Observe that if $(A, B; X, Y) = -1$, then $(A, B; Y, X) = (B, A; X, Y) = -1$. We sometimes also say that Y is the **harmonic conjugate** of X with respect to $\overline{AB}$; as the name suggests, it is unique, and the harmonic conjugate of Y is X itself.

Harmonic bundles are important because they appear naturally in many configurations. We present four configurations in which they arise.

The first lemma is trivial to prove, but gives us a new way to handle midpoints, particularly if they appear along with parallel lines.

Lemma 9.8 (Midpoints and Parallel Lines). *Given points A and B, let M be the midpoint of $\overline{AB}$ and P_∞ the point at infinity of line AB. Then $(A, B; M, P_\infty)$ is a harmonic bundle.*

The next lemma (illustrated in Figure 9.3A) describes harmonic quadrilaterals in terms of tangents to a circle.

Lemma 9.9 (Harmonic Quadrilaterals). *Let ω be a circle and let P be a point outside it. Let $\overline{PX}$ and $\overline{PY}$ be tangents to ω. Take a line through P intersecting ω again at A and B. Then*

(a) $AXBY$ is a harmonic quadrilateral.
(b) If $Q = \overline{AB} \cap \overline{XY}$, then $(A, B; Q, P)$ is a harmonic bundle.

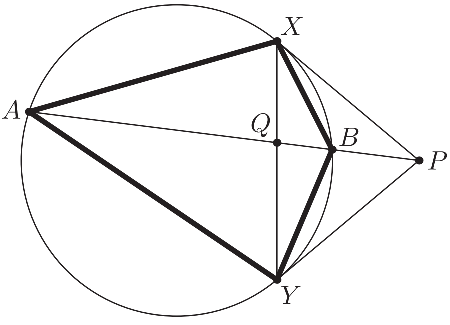

Figure 9.3A. A harmonic quadrilateral. $(A, B; P, Q)$ is also harmonic.

Proof. We use symmedians. We obtain $\frac{XA}{XB} = \frac{YA}{YB}$ from Lemma 4.26, and $(A, B; X, Y)$ is negative by construction. This establishes that $AXBY$ is harmonic.

To see that $(A, B; Q, P)$ is harmonic, just write

$$(A, B; X, Y) \stackrel{X}{=} (A, B; Q, P).$$

Here we are projecting from the circle onto the line AB from X, noting that line XX in this context is actually just the tangent to ω. (To see this, consider the behavior of line XX' when X' is very close to X on the circle.) $\square$

This also implies the tangents to A and B intersect on line XY. (Why?)

An important special case is when $\overline{AB}$ is selected as a diameter of ω. In that case, P and Q are inverses when inverting around ω. In full detail, we have the following.

Proposition 9.10 (Inversion Induces Harmonic Bundles). *Let P be a point on line $\overline{AB}$, and let P^* denote the image of P after inverting around the circle with diameter $\overline{AB}$. Then $(A, B; P, P^*)$ is harmonic.*

The third and fourth lemmas involve no circles at all. Actually the fourth is really a consequence of the third.

Lemma 9.11 (Cevians Induces Harmonic Bundles). *Let ABC be a triangle with concurrent cevians $\overline{AD}$, $\overline{BE}$, $\overline{CF}$ (possibly on the extensions of the sides). Line EF meets BC at X (possibly at a point at infinity). Then $(X, D; B, C)$ is a harmonic bundle.*

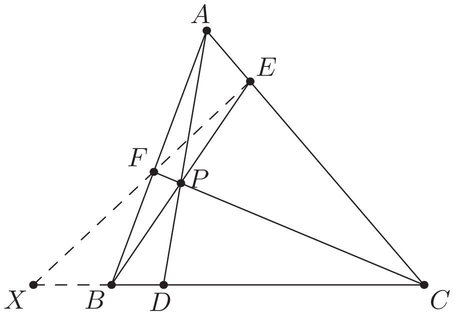

Figure 9.3B. Ceva's and Menelaus's theorems produce $(X, D; B, C) = -1$.

Proof. Use the directed form Ceva's theorem and Menelaus's theorem on Figure 9.3B. □

Lemma 9.12 (Complete Quadrilaterals Induces Harmonic Bundles). *Let $ABCD$ be a quadrilateral whose diagonals meet at K. Lines AD and BC meet at L, and line KL meets $\overline{AB}$ and $\overline{CD}$ at M and N. Then $(K, L; M, N)$ is a harmonic bundle.*

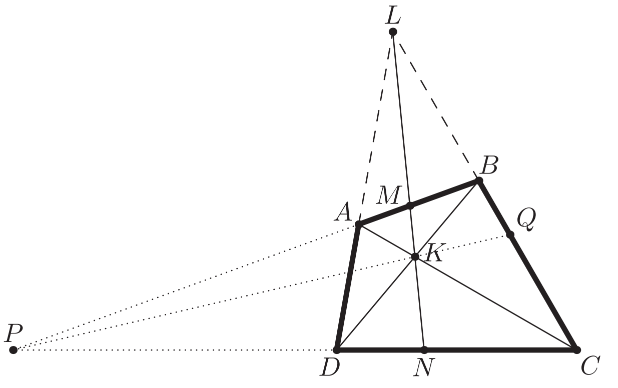

Figure 9.3C. You can modify Lemma 9.11 to get $(K, L; M, N)$ a harmonic bundle as well.

Proof. As in Figure 9.3C, let $P = \overline{AB} \cap \overline{CD}$, and let $Q = \overline{PK} \cap \overline{BC}$. By Lemma 9.11, $(Q, L; B, C) = -1$. Projecting onto the desired line, we derive

$$-1 = (Q, L; B, C) \stackrel{P}{=} (K, L; M, N).$$ □

Harmonic bundles let us move from one of these configurations to the others. As an example, we revisit Problem 4.45.

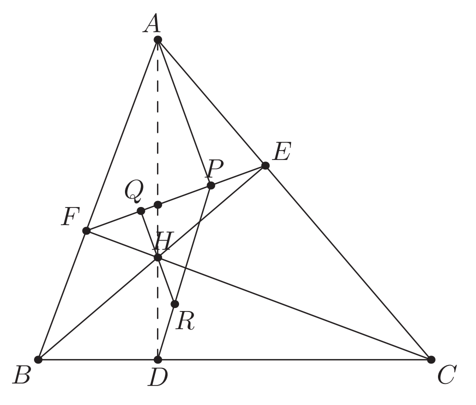

Figure 9.3D. The first problem from the USA TST 2011.

Example 9.13 (USA TST 2011/1). In an acute scalene triangle ABC, points D, E, F lie on sides BC, CA, AB, respectively, such that $\overline{AD} \perp \overline{BC}$, $\overline{BE} \perp \overline{CA}$, $\overline{CF} \perp \overline{AB}$.

Altitudes $\overline{AD}$, $\overline{BE}$, $\overline{CF}$ meet at orthocenter H. Points P and Q lie on segment $\overline{EF}$ such that $\overline{AP} \perp \overline{EF}$ and $\overline{HQ} \perp \overline{EF}$. Lines DP and QH intersect at point R. Compute HQ/HR.

We might readily dismiss this as an uninteresting problem. The answer is 1; the problem is just Lemma 4.9 applied to triangle DEF. However, it turns out there is a quick projective proof completely independent of this.

Remember Lemma 9.8? We indeed have both a midpoint (H of $\overline{QR}$) and a line parallel to it ($\overline{AP} \parallel \overline{QR}$). Hence we take perspectivity through P. More precisely, let P_∞ be the point at infinity for $\overline{AP}$ and $\overline{QR}$. Then

$$(Q, R; H, P_\infty) \stackrel{P}{=} (\overline{QP} \cap \overline{AD}, D; H, A).$$

If we can show the latter is a harmonic bundle, then we are done. But this is just Lemma 9.12!

Needless to say, we can go backwards, as in the proof below.

Solution. By Lemma 9.12, $(A, H; \overline{AD} \cap \overline{EF}, D) = -1$. Projecting through P, we find $(P_\infty, H; Q, R) = -1$, where P_∞ is the point at infinity on parallel lines AP and QR. Hence $\frac{HQ}{HR} = 1$. □

Problems for this Section

Problem 9.14. Check the details in the proofs of Lemma 9.11 and Lemma 9.18.

Problem 9.15. In the coordinate plane, the points $A = (-1, 0)$, $B = (1, 0)$, $X = \left(\frac{1}{100}, 0\right)$ and $Y = (m, 0)$ form a harmonic bundle $(A, B; X, Y) = -1$. What is m? **Hint**: 334

Problem 9.16. Show that Problem 1.43 (see Figure 9.3E) is immediate from the tools developed in this chapter. **Hints**: 107 687 607 451 520

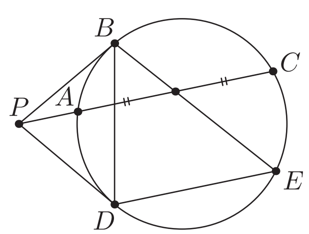

Figure 9.3E. Solve JMO 2011/5 (Problem 1.43) using harmonic bundles.

Lemma 9.17 (Midpoint Lengths). *Points A, X, B, P lie on a line in that order, and $(A, B; X, P) = -1$. Let M be the midpoint of $\overline{AB}$. Show that $MX \cdot MP = \left(\frac{1}{2}AB\right)^2$ and $PX \cdot PM = PA \cdot PB$.* **Hints**: 41 557

9.4 Apollonian Circles

There is one additional configuration with naturally occurring harmonic bundles. First, we need to state a lemma (see Figure 9.4A).

Lemma 9.18 (Right Angles and Bisectors). *Let X, A, Y, B be collinear points in that order and let C be any point not on this line. Then any two of the following conditions implies the third condition.*

(i) $(A, B; X, Y)$ *is a harmonic bundle.*
(ii) $\angle XCY = 90^\circ$.
(iii) $\overline{CY}$ *bisects* $\angle ACB$.

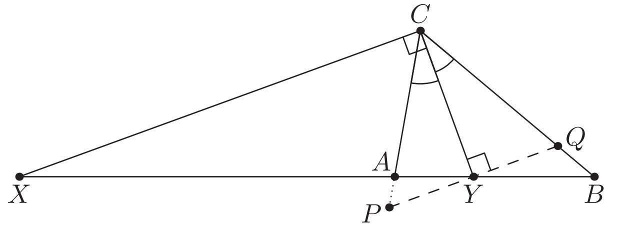

Figure 9.4A. $\overline{CX}$ and $\overline{CY}$ are external and internal angle bisectors.

Proof. There is a straightforward trigonometric proof, but here we present a synthetic solution. Draw the line through Y parallel to $\overline{CX}$ and let it intersect rays CA and CB at P and Q, respectively. Since $\triangle XAC \sim \triangle YAP$ and $\triangle XBC \sim \triangle YBQ$, we derive

$$PY = \frac{AY}{AX} \cdot CX \text{ and } QY = \frac{BY}{BX} \cdot CX.$$

Thus $PY = QY$ if and only if $(A, B; X, Y) = -1$. Now any two of the conditions imply $\triangle CYP$ and $\triangle CYQ$ are congruent, which gives the third. □

While this is useful in its own right, it leads directly to the so-called **Apollonian circle**, which is a way of linking angles with ratios. The statement is as follows.

Theorem 9.19 (Apollonian Circles). *Let $\overline{AB}$ be a segment and $k \neq 1$ be a positive real. The locus of points C satisfying $\frac{CA}{CB} = k$ is a circle whose diameter lies on $\overline{AB}$.*

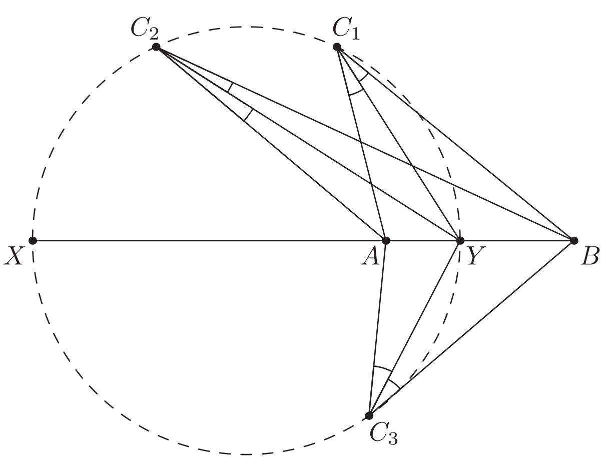

Figure 9.4B. Apollonian Circles

This is really just a restatement of Lemma 9.18, with the congruent angles rewritten as a ratio because of the angle bisector theorem. Here are the details; refer to Figure 9.4B.

Proof. First of all, let X and Y be the two points on line AB with

$$\frac{XA}{XB} = \frac{YA}{YB} = k.$$

Without loss of generality, Y lies on $\overline{AB}$.

Now observe that for any other point C, $\frac{CA}{CB} = k$ is just equivalent to $\angle CAY = \angle YBC$ by the angle bisector theorem. That is equivalent to $\angle XCY = 90^\circ$ by Lemma 9.18, and hence we discover the Apollonian circle. □

Problems for this Section

Problem 9.20. In the notation of Figure 9.4B, what is the Apollonian circle of $\overline{XY}$ through A? **Hints**: 411 70

Problem 9.21. Check that as k varies, the resulting set of circles are all coaxial*. **Hints**: 315 147

Lemma 9.22 (Harmonic Bundles on the Bisector). *Let ABC be a triangle with incenter I and A-excenter I_A. Prove that*

$$(I, I_A; A, \overline{AI} \cap \overline{BC}) = -1.$$

9.5 Poles/Polars and Brocard's Theorem

Projective and inversive techniques are actually closely related by the concepts of poles and polars.

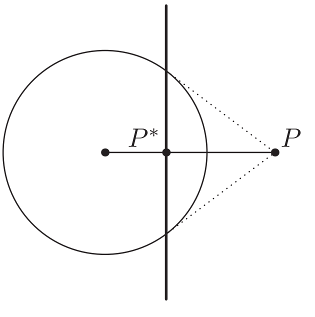

Figure 9.5A. The polar of point P is the line shown.

Fix a circle ω with center O and a point P. Let P^* be the inverse of P with respect to inversion around ω. The **polar** of point P (possibly at infinity and distinct from O) is the line passing through P^* perpendicular to $\overline{OP}$. As we have mentioned before, when P is outside circle ω then its polar is the line through the two tangency points from P to ω. The polar of O is just the line at infinity.

* Actually, it turns out any non-intersecting coaxial circles are Apollonian.

Similarly, given a line ℓ not through O, we define its **pole**† as the point P that has ℓ as its polar.

First, an obvious result that is nonetheless useful.

Theorem 9.23 (La Hire's Theorem). *A point X lies on the polar of a point Y if and only if Y lies on the polar of X.*

Proof. Left as an exercise. It is merely similar triangles. □

La Hire's theorem demonstrates a concept called **duality**: one can exchange points for lines, lines for intersections, collinearity for concurrence. Simply swap every point with its polar and every line with its pole.

We can now state an important result relating poles and polars to harmonic bundles.

Proposition 9.24. *Let $\overline{AB}$ be a chord of a circle ω and select points P and Q on line AB. Then $(A, B; P, Q) = -1$ if and only if P lies on the polar of Q.*

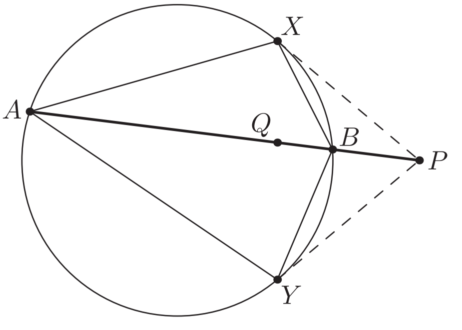

Figure 9.5B. Harmonic quadrilaterals again.

Proof. We consider only the case where P is outside ω and Q is inside it. Construct the tangents $\overline{PX}$ and $\overline{PY}$ to ω. Lemma 9.9 gives

$$(A, B; P, \overline{XY} \cap \overline{AB}) = -1,$$

so Q lies on the polar of P (namely line XY) if and only if $(A, B; P, Q) = -1$. □

We are now ready to state one of the most profound theorems about cyclic quadrilaterals. It shows that any cyclic quadrilateral has hidden within it three pairs of poles and polars.

Theorem 9.25 (Brocard's Theorem). *Let $ABCD$ be an arbitrary cyclic quadrilateral inscribed in a circle with center O, and set $P = \overline{AB} \cap \overline{CD}$, $Q = \overline{BC} \cap \overline{DA}$, and $R = \overline{AC} \cap \overline{BD}$. Then P, Q, R are the poles of QR, RP, PQ, respectively.*

In particular, O is the orthocenter of triangle PQR.

We say that triangle PQR is **self-polar** with respect to ω, because each of its sides is the polar of the opposite vertex.

† Not the best choice of terms, as the two are easily confused. Mnemonic: "pole" is shorter than "polar", and points are much smaller than lines.

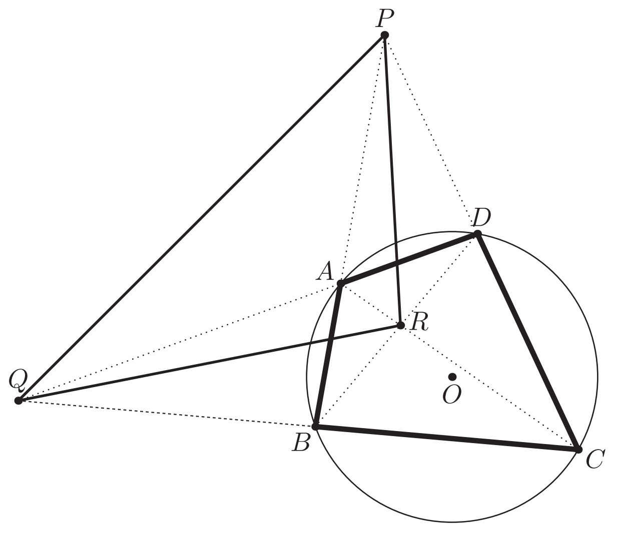

Figure 9.5C. The triangle PQR determined by completing a cyclic quadrilateral is self-polar.

Take a moment to appreciate the power of Brocard's theorem. Nowhere do the words "pole", "polar", "harmonic", "projective", or anything of that sort appear in the hypothesis. We could have stated this theorem in Chapter 1—all we did was take a completely arbitrary cyclic quadrilateral and intersect the sides and diagonals—and then suddenly, we have an entire orthocenter! It seems too good to be true. This really highlights the type of problems that projective geometry handles well: anything with lots of intersections and maybe a few circles.

On to the proof of the theorem. The idea is that Brocard's theorem looks a lot like Lemma 9.11.

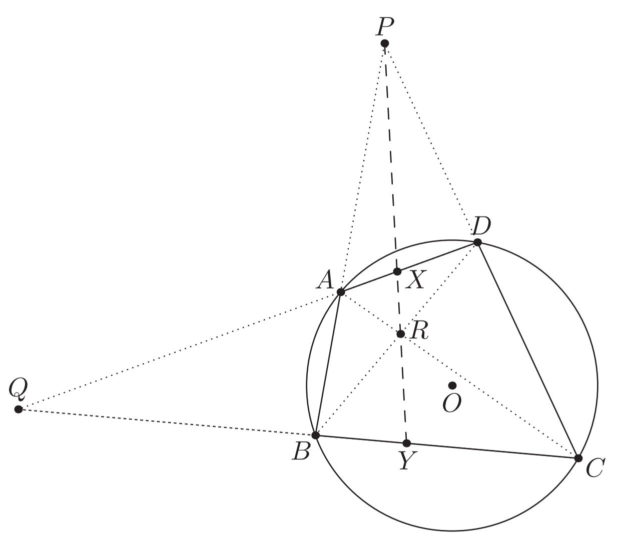

Figure 9.5D. Triangle PQR is self-polar.

Proof. First, we show that Q is the pole of line PR. Define the points $X = \overline{AD} \cap \overline{PR}$ and $Y = \overline{BC} \cap \overline{PR}$, as in Figure 9.5D. By Lemma 9.11, both $(A, D; Q, X)$ and $(B, C; Q, Y)$ are harmonic bundles.

Therefore, X and Y both lie on the polar of Q, by Proposition 9.24. Since the polar of Q is a line, it must be precisely line XY, which is the same as line PR.

The same can be used to show that P is the pole of line QR and R is the pole of line PQ; projective geometry is immune to configuration issues. (This is part of the reason we like points at infinity.) This gives that PQR is indeed self-polar. Finally, the definition of a polar implies that O is the orthocenter of triangle PQR, completing the proof. $\square$

Problems for this Section

Problem 9.26. Prove La Hire's theorem (Theorem 9.23).

Lemma 9.27 (Self-Polar Orthogonality). *Let ω be a circle and suppose P and Q are points such that P lies on the pole of Q (and hence Q lies on the pole of P). Prove that the circle γ with diameter $\overline{PQ}$ is orthogonal to ω.* **Hint**: 616

Problem 9.28. Let ABC be an acute scalene triangle, and let H be a point inside it such that $\overline{AH} \perp \overline{BC}$. Rays BH and CH meet $\overline{AC}$ and $\overline{AB}$ at E, F. Prove that if quadrilateral $BFEC$ is cyclic then H is in fact the orthocenter of ABC. **Hints**: 492 52

9.6 Pascal's Theorem

Pascal's theorem is of a different flavor than the previous theorems, but is useful in similar situations. It handles many points on a circle and their intersections. Here is the statement[‡]; see Example 7.27 for a proof. Many other proofs exist, of course.

Theorem 9.29 (Pascal's Theorem). *Let $ABCDEF$ be a cyclic hexagon, possibly self-intersecting. Then the points $\overline{AB} \cap \overline{DE}$, $\overline{BC} \cap \overline{EF}$, and $\overline{CD} \cap \overline{FA}$ are collinear.*

Note that Pascal's theorem can look very different depending on what order the vertices lie in. Figure 9.6A shows four different shapes that Pascal's theorem can take on. It is often useful to take two consecutive vertices of the hexagon to be the same point. The "side" AA degenerates to a tangent to the circle at A.[§] An example of this technique is in the solution to Example 9.38.

For an example, we revisit the first part of Lemma 4.40, and give a short proof using Pascal's theorem.

Example 9.30. Let ABC be a triangle inscribed in a circle. The A-mixtilinear circle is drawn, tangent to $\overline{AB}$, $\overline{AC}$ at K, L. Then the incenter I is the midpoint of $\overline{KL}$.

[‡] The converse is also true if we replace "circle" with "conic". See the next section on projective transformations.

[§] Think of it this way: $\overline{XY}$ is the line intersecting the circle at points X and Y. So $\overline{AA}$ is a line intersecting the circle at A and A, i.e., the tangent to A.

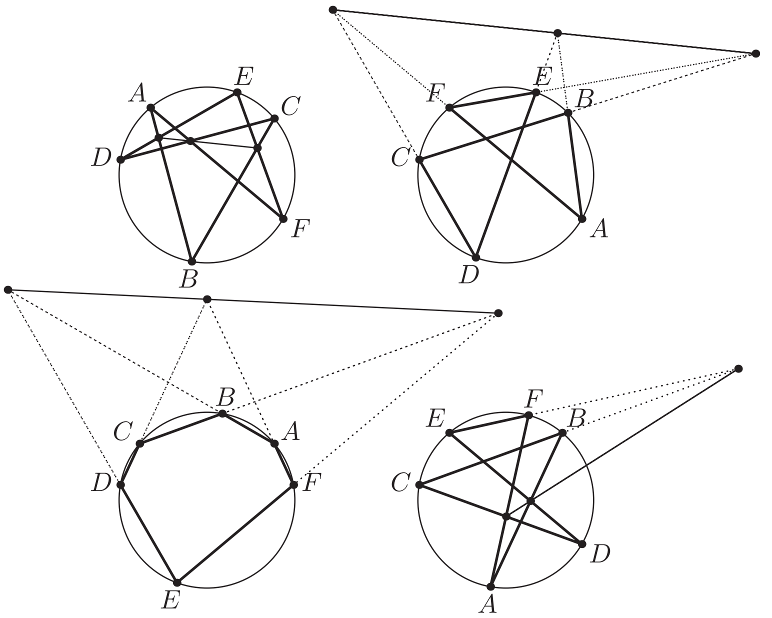

Figure 9.6A. The many faces of Pascal's theorem.

Proof. Obviously $\overline{AI}$ bisects $\overline{KL}$ (since $AK = AL$ and $\angle KAI = \angle IAL$) so it suffices to prove that K, I, L are collinear.

By Lemma 4.33, M_C, K, T are collinear, where M_C is the midpoint of arc AB not containing C. In particular, C, I, M_C are collinear. Similarly, the midpoint M_B of arc AC lies on both lines BI and LT. Now we just apply Pascal's theorem on the hexagon ABM_BTM_CC. $\square$

An even more striking illustration is Problem 9.32 below.

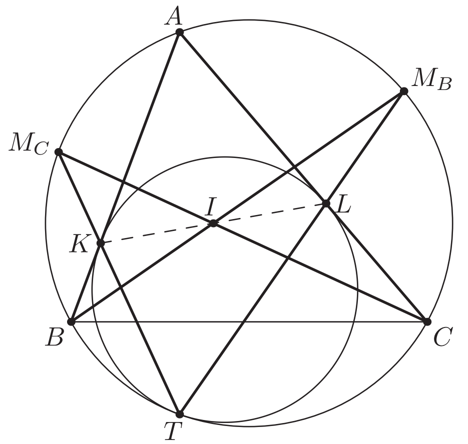

Figure 9.6B. Using Pascal's theorem on the A-mixtilinear incircle.

Problems for this Section

Problem 9.31. Let ABC be a triangle with circumcircle Γ. Let X be the intersection of line BC with the tangent to Γ at A. Define Y and Z similarly. Show that X, Y, Z are collinear. **Hint**: 378

Problem 9.32. Let $ABCD$ be a cyclic quadrilateral and apply Pascal's theorem to $AABCCD$ and $ABBCDD$. What do we discover? **Hints**: 421 473 309

9.7 Projective Transformations

This is only a brief digression on what is otherwise a deep topic. See the last chapter of [7] for further exposition.

Occasionally we run into a problem that we say is *purely projective*. Essentially this means the problem statement involves only intersections, tangency, and perhaps a few circles. This happens very rarely, but when it does, the problems can usually be eradicated via **projective transformations**.

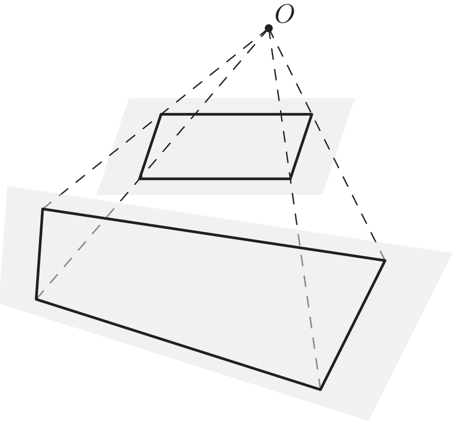

Figure 9.7A. An example of a projective transformation.

Projective transformations are essentially the most general type of transformation. Actually, they are defined as any map that sends lines to lines and conics to conics (but need not preserve anything else). Loosely speaking, a **conic** is a second-degree curve in the plane determined by five points. In more precise terms, a conic is a curve in the xy-plane of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

extended to include points at infinity. This includes parabolas, hyperbolas, and ellipses (in particular, circles). For our purposes, we only care that a circle is a conic. See Figure 9.7A.

Why would we consider a transformation that preserves so few things? The gain is encapsulated in the following theorem, stated without proof, which exploits the generality of the transformation.

Theorem 9.33 (Projective Transformations). *Each of the following is achievable with a unique projective transformation.*

(a) Taking four points A, B, C, D (no three collinear) to any other four points W, X, Y, Z (no three collinear).
(b) Taking a circle to itself and a point P inside the circle to any other point Q inside the circle.
(c) Taking a circle to itself and any given line outside the circle into the line at infinity.

Furthermore, projective transformations preserve the cross ratio of any four collinear points. Moreover, if four concyclic points are sent to four concyclic points, then the cross ratio of the quadrilaterals are the same.

The power of this technique is made most clear by example.

Example 9.34. Let $ABCD$ be a quadrilateral. Define the points $P = \overline{AD} \cap \overline{BC}$, $Q = \overline{AB} \cap \overline{CD}$, and $R = \overline{AC} \cap \overline{BD}$. Let X_1, X_2, Y_1, Y_2 denote $\overline{PR} \cap \overline{AD}$, $\overline{PR} \cap \overline{BC}$, $\overline{QR} \cap \overline{AB}$, $\overline{QR} \cap \overline{CD}$.

Prove that lines X_1Y_1, X_2Y_2, and PQ are concurrent.

This problem looks like a nightmare until we realize that it is purely projective. That means we can make some very convenient assumptions—we simply use a projective map taking $ABCD$ to a square $A'B'C'D'$.

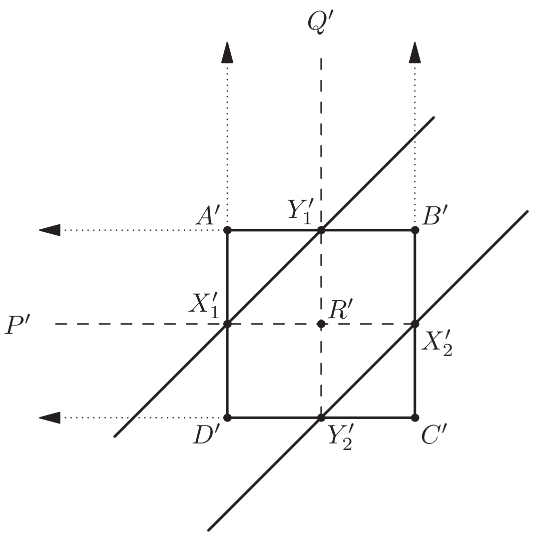

Figure 9.7B. We can take $ABCD$ to a square, trivializing the problem.

Solution. By Theorem 9.33, we can use a projective transformation to send $ABCD$ to the vertices of a square $A'B'C'D'$. Then P' is the intersection of lines $A'D'$ and $B'C'$, since projective transformations preserve intersections. We can define the remaining points similarly.

The problem is now trivial: just look at Figure 9.7B! P' and Q' become the points at infinity, and we find that X_1', X_2', Y_1', Y_2' are just midpoints of the respective sides. Hence the intersection of lines $\overline{X_1'Y_1'}$ and $\overline{X_2'Y_2'}$ is yet another point at infinity (as they are parallel). This implies P', Q', and $\overline{X_1'Y_1'} \cap \overline{X_2'Y_2'}$ are collinear along the line at infinity. $\square$

We can even extend this technique to tackle problems that do not look purely projective when the condition can be re-written with cross ratios. For example, consider the famous butterfly theorem.

Theorem 9.35 (Butterfly Theorem). *Let $\overline{AB}$, $\overline{CD}$, $\overline{PQ}$ be chords of a circle concurrent at M. Put $X = \overline{PQ} \cap \overline{AD}$ and $Y = \overline{PQ} \cap \overline{BC}$. If $MP = MQ$ then $MX = MY$.*

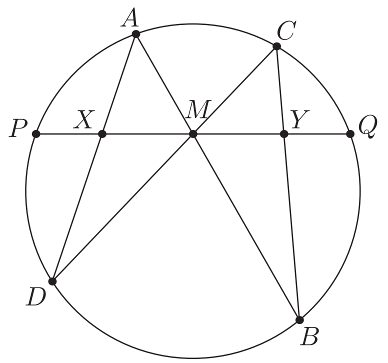

Figure 9.7C. The butterfly theorem.

Proof. This problem looks completely projective except for the midpoint condition. We can handle this by adding the point at infinity P_∞ to line PQ. The condition becomes $(P, Q; P_\infty, M) = -1$, and we wish to show that $(X, Y; P_\infty, M) = -1$.

By rewriting the givens as cross ratios, the problem becomes purely projective! We therefore take the projective transformation sending M to the center of the circle, say M'. Then $\overline{P'Q'}$ is a diameter. Because we must have the cross ratio $(P', Q', P'_\infty, M') = -1$ is preserved, we find that P'_∞ is still the point at infinity. Hence it simply suffices to prove that M' is the midpoint of $\overline{X'Y'}$.

On the other hand, proving the butterfly theorem when M is the center of the circle is not very hard. Actually, it is obvious by symmetry. Therefore $(X', Y', P'_\infty, M') = -1$. Consequently $(X, Y; P_\infty, M) = -1$ as well and we are done. $\square$

Problems for this Section

Problem 9.36. Give a short proof of Lemma 9.9 using projective transformations. **Hints**: 183 218 231

Problem 9.37. Give a short proof of Lemma 9.11 using projective transformations. **Hints**: 333 595

9.8 Examples

We present two example problems. First, let us consider the following problem from the 51st IMO.

Example 9.38 (IMO 2010/2). Let I be the incenter of a triangle ABC and let Γ be its circumcircle. Let line AI intersect Γ again at D. Let E be a point on arc $\widehat{BDC}$ and F a point on side BC such that $\angle BAF = \angle CAE < \frac{1}{2}\angle BAC$. Finally, let G be the midpoint of $\overline{IF}$. Prove that $\overline{DG}$ and $\overline{EI}$ intersect on Γ.

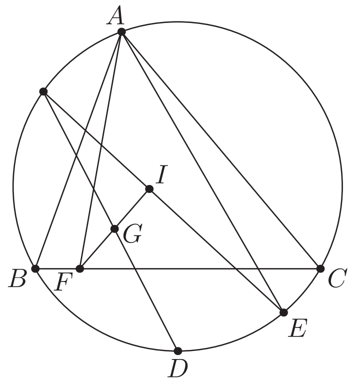

Figure 9.8A. Example 9.38.

We begin by extending $\overline{AF}$ to meet Γ again at a point F_1; evidently $\overline{F_1E} \parallel \overline{BC}$. We also let K denote the second intersection of $\overline{EI}$ with Γ. Our goal is to prove that $\overline{DK}$ bisects $\overline{IF}$.

Seeing so many points and intersections on a circle motivates us to try Pascal's theorem in the hopes of finding something interesting. Specifically, we have $I = \overline{AD} \cap \overline{KE}$, $\overline{DD} \cap \overline{EF_1}$ is the point at infinity, and $F = \overline{AF_1} \cap \overline{BC}$. Trying to string two of these into one application of Pascal's theorem, we find with some trial and error that the hexagon AF_1EKDD is useful.

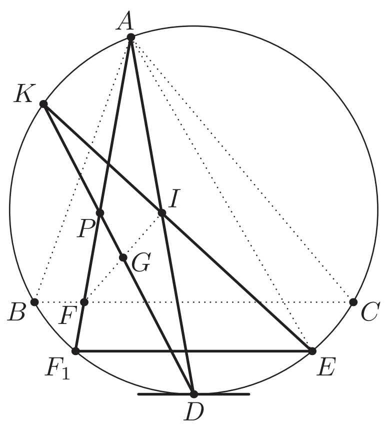

Figure 9.8B. Applying Pascal's theorem on Example 9.38.

Pascal's theorem now implies that $\overline{AF_1} \cap \overline{KD}$, the point at infinity $\overline{F_1E} \cap \overline{DD}$, and the incenter $I = \overline{DA} \cap \overline{KE}$ are collinear. In other words, if we set $P = \overline{AF_1} \cap \overline{KD}$, then we find that $\overline{IP} \parallel \overline{EF_1} \parallel \overline{BC}$.

Once the point P is introduced, we can effectively ignore the points E, F_1, and K now. In other words, we have the convenient recasting of the problem as follows.

> Let $\overline{AF}$ be a cevian of the triangle ABC and let P be a point on $\overline{AF}$ with $\overline{IP} \parallel \overline{BC}$. If D is the midpoint of arc $\widehat{BC}$ not containing A, then $\overline{DP}$ bisects $\overline{IF}$.

This is much simpler, and you can actually finish using barycentric coordinates. At least this indicates that we are probably on the right track. So what do we do next?

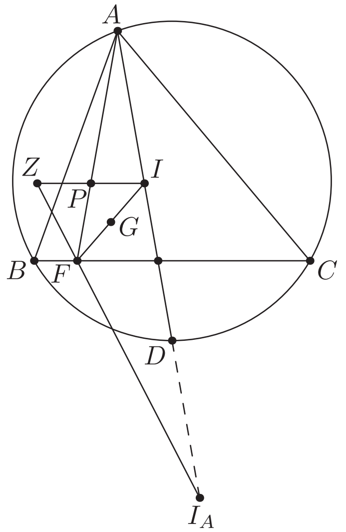

Figure 9.8C. The finishing touch using harmonic bundles.

Seeing the midpoint, we consider a homothety at I with ratio 2, which conveniently grabs the excenter I_A. That means it suffices to prove that if $Z = \overline{I_A F} \cap \overline{IP}$, then P is the midpoint of $\overline{IZ}$. Seeing midpoints and parallel lines once again, we take harmonic bundles (in light of Lemma 9.8). And indeed, the first decent choice of a point on $\overline{BC}$ works; perspectivity at F solves the problem.

Solution to Example 9.38. Let $\overline{EI}$ meet Γ again at K and $\overline{AF}$ meet Γ again at F_1. Set $P = \overline{DK} \cap \overline{AF}$ and $Z = \overline{IP} \cap \overline{I_A F}$. By Pascal's theorem on AF_1EKDD, we see that $\overline{IP} \parallel \overline{BC}$.

Setting I_A as the A-excenter and recalling Lemma 9.22 gives

$$-1 = \left(I, I_A; A, \overline{AI} \cap \overline{BC}\right) \stackrel{F}{=} \left(I, Z; P, \overline{BC} \cap \overline{IP}\right).$$

Since $\overline{IP} \parallel \overline{BC}$, we conclude that P is the midpoint of $\overline{IZ}$. Then we simply take a homothety at I. $\square$

Our other example is the final problem from an Asian-Pacific olympiad; it yields many different projective solutions. We present three of them.

Example 9.39 (APMO 2013/5). Let $ABCD$ be a quadrilateral inscribed in a circle ω, and let P be a point on the extension of $\overline{AC}$ such that $\overline{PB}$ and $\overline{PD}$ are tangent to ω. The tangent at C intersects $\overline{PD}$ at Q and the line AD at R. Let E be the second point of intersection between $\overline{AQ}$ and ω. Prove that B, E, R are collinear.

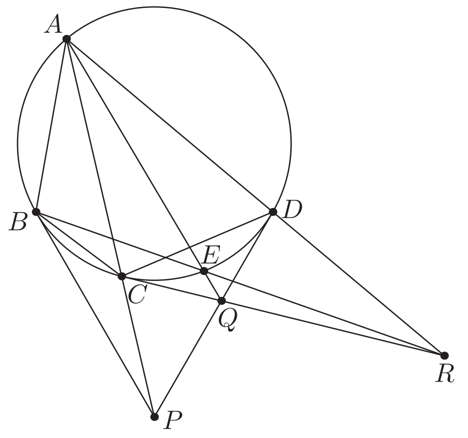

Figure 9.8D. Problem 5 from APMO 2013.

We immediately recognize Lemma 9.9 twice: $ACED$ and $ABCD$ are both harmonic quadrilaterals. This motivates us to try projective geometry in the first place, since there are a lot of intersections and the conditions are natural in the language of harmonic bundles.

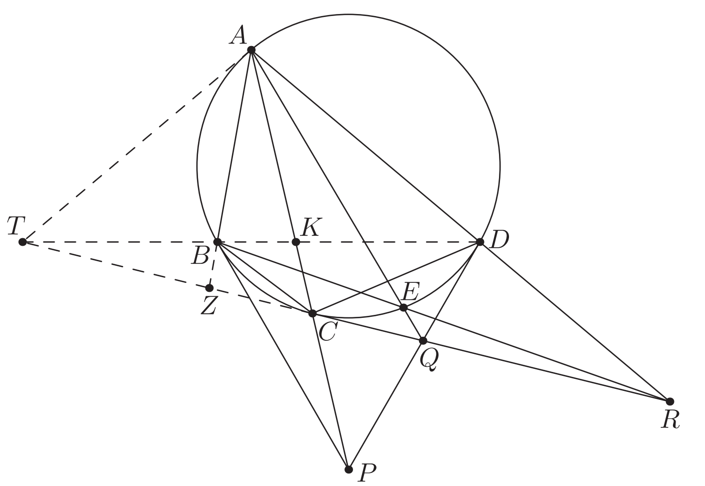

Figure 9.8E. A solution to Example 9.39 that involves only harmonic bundles.

In order to place things more in the frame of our projective tools, we let E' be the second intersection of line BR and ω. Then it would just suffice to prove $ACE'D$ is harmonic (rather than prove three points are collinear). How might we do that? We wish to prove that $(A, E'; C, D) = -1$. Are there any points that look good for projecting through on ω? After some trial we find that B looks like a good choice, because it handles the other points somewhat nicely, but more importantly it lets us deal with the point E'.

Because we again want to focus on making point E' behave well, we choose to project onto line CR.

So we find that

$$(A, E'; C, D) \stackrel{B}{=} (\overline{AB} \cap \overline{CR}, R; C, \overline{BD} \cap \overline{CR}).$$

Taking advantage of the fact that $ABCD$ is harmonic, we put $T = \overline{BD} \cap \overline{CR}$ as the intersection of the tangents at A and C (hence on line BD). The point T seems nice because it is pretty closely tied to $ABCD$.

On the other hand we should probably clean up $\overline{AB} \cap \overline{CR}$ in the next projection. Since we already took perspectivity from B, we try taking perspectivity from A this time (otherwise we are back where we started). Now the most logical choice for the line to project onto is BD. Letting $Z = \overline{AB} \cap \overline{CR}$ for brevity, we find

$$(Z, R; C, T) \stackrel{A}{=} (B, D; \overline{AC} \cap \overline{BD}, T).$$

But this is harmonic by Lemma 9.9. Hence with just two projections we are done.

Solution 1. Set $T = \overline{BD} \cap \overline{CR}$, $K = \overline{AC} \cap \overline{BD}$, $Z = \overline{AB} \cap \overline{CR}$ and let E' be the second intersection of $\overline{BR}$ with ω. Since $ABCD$ is harmonic, we have T, K, B, D collinear and therefore

$$-1 = (T, K; B, D) \stackrel{A}{=} (T, C; Z, R) \stackrel{B}{=} (D, C; A, E').$$

But $DACE$ is harmonic, so $E = E'$. □

A second solution involves interpreting the problem from the context of symmedians (see Lemma 4.26). We can view $\overline{DB}$ and $\overline{AE}$ as the symmedians of triangle ACD. Suddenly we can ignore the points P and Q completely! On the other hand we should probably add in the symmedian point K of triangle ACD, which is the intersection of $\overline{AE}$ and $\overline{BD}$.

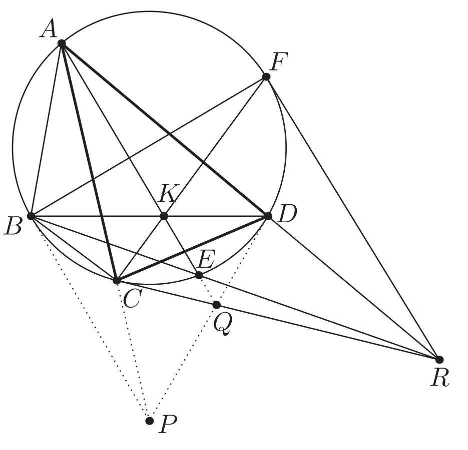

Figure 9.8F. Solving Example 9.39 using symmedians.

Now what of the point R? It is the intersection of the tangent at C with line AD. Trying to complete Lemma 9.9 again, we let F be the other point on ω other than C such that $\overline{RF}$

is a tangent. Hence $ACDF$ is harmonic. So $\overline{CF}$ is a symmedian as well. This completes the picture of the symmedian point. In particular, K lies on $\overline{CF}$.

Now for the finish. By Brocard's theorem, $\overline{BE} \cap \overline{AD}$ is the point on $\overline{AD}$ that lies on the polar of $K = \overline{BD} \cap \overline{AE}$. This is none other than the point R.

Solution 2. Let $K = \overline{AE} \cap \overline{BD}$ be the symmedian point of triangle ACD. Let F be the second intersection of ray CK with (ACD). Noticing the symmedians, we find three harmonic quadrilaterals $ACED$, $ABCD$, and $ACDF$.

In harmonic quadrilateral $ACDF$, we notice (by Lemma 9.9, say), that R is the pole of $\overline{CF}$. Because $\overline{CF}$ contains K, point R lies on the polar of K. Now by Brocard's theorem, the intersection of line BE with $\overline{AD}$ lies on the polar of K as well, implying that B, E, R are collinear. □

Finally, one last solution—note this problem is purely projective!

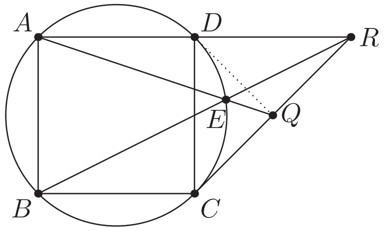

Figure 9.8G. Projective transformations trivialize Example 9.39, because they allow us to assume $ABCD$ is a square.

Take a projective transformation that fixes ω and sends the point $\overline{AC} \cap \overline{BD}$ to the center of the circle. Thus $ABCD$ is a rectangle. Because $ABCD$ is harmonic, it must in fact be a square. Thus P is the point at infinity along $\overline{AB} \parallel \overline{CD}$ and the problem is not very hard now.

9.9 Problems

Lemma 9.40 (Incircle Polars). *Let ABC be a triangle with contact triangle DEF and incenter I. Lines EF and BC meet at K. Prove that $\overline{IK} \perp \overline{AD}$.* **Hints**: 351 689 **Sol:** p.275

Theorem 9.41 (Desargues' Theorem). *Let ABC and XYZ be triangles in the projective plane. We say that the two triangles are* ***perspective from a point*** *if lines $\overline{AX}$, $\overline{BY}$, and $\overline{CZ}$ concur (possibly at infinity), and we say they are* ***perspective from a line*** *if the points $\overline{AB} \cap \overline{XY}$, $\overline{BC} \cap \overline{YZ}$, $\overline{CA} \cap \overline{ZX}$ are collinear. Prove that these two conditions are equivalent.* **Hints**: 253 456

Problem 9.42 (USA TSTST 2012/4). In scalene triangle ABC, let the feet of the perpendiculars from A to $\overline{BC}$, B to $\overline{CA}$, C to $\overline{AB}$ be A_1, B_1, C_1, respectively. Denote by A_2 the intersection of lines BC and B_1C_1. Define B_2 and C_2 analogously. Let D, E, F be the respective midpoints of sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$. Show that the perpendiculars from D to $\overline{AA_2}$, E to $\overline{BB_2}$, and F to $\overline{CC_2}$ are concurrent. **Hints**: 308 233

Problem 9.43 (Singapore TST). Let ω and O be the circumcircle and circumcenter of right triangle ABC with $\angle B = 90^\circ$. Let P be any point on the tangent to ω at A other than A, and suppose ray PB intersects ω again at D. Point E lies on line CD such that $\overline{AE} \parallel \overline{BC}$. Prove that P, O, and E are collinear. **Hints**: 587 675

Problem 9.44 (Canada 1994/5). Let ABC be an acute triangle. Let $\overline{AD}$ be the altitude on $\overline{BC}$, and let H be any interior point on $\overline{AD}$. Lines BH and CH, when extended, intersect $\overline{AC}$, $\overline{AB}$ at E and F respectively. Prove that $\angle EDH = \angle FDH$. **Hints**: 20 164 80 **Sol:** p.275

Problem 9.45 (Bulgarian Olympiad 2001). Let ABC be a triangle and let k be a circle through C tangent to $\overline{AB}$ at B. Side $\overline{AC}$ and the C-median of $\triangle ABC$ intersect k again at D and E, respectively. Prove that if the intersecting point of the tangents to k through C and E lies on the line BD then $\angle ABC = 90^\circ$. **Hints**: 111 318 571

Problem 9.46 (ELMO Shortlist 2012). Let ABC be a triangle with incenter I. The foot of the perpendicular from I to $\overline{BC}$ is D, and the foot of the perpendicular from I to $\overline{AD}$ is P. Prove that $\angle BPD = \angle DPC$. **Hints**: 240 354 347 **Sol:** p.276

Problem 9.47 (IMO 2014/4). Let P and Q be on segment BC of an acute triangle ABC such that $\angle PAB = \angle BCA$ and $\angle CAQ = \angle ABC$. Let M and N be the points on AP and AQ, respectively, such that P is the midpoint of AM and Q is the midpoint of AN. Prove that the intersection of BM and CN is on the circumference of triangle ABC. **Hints**: 145 216 286 **Sol:** p.276

Problem 9.48 (Shortlist 2004/G8). Given a cyclic quadrilateral $ABCD$, let M be the midpoint of the side CD, and let N be a point on the circumcircle of triangle ABM. Assume that the point N is different from the point M and satisfies $\frac{AN}{BN} = \frac{AM}{BM}$. Prove that the points E, F, N are collinear, where $E = \overline{AC} \cap \overline{BD}$ and $F = \overline{BC} \cap \overline{DA}$. **Hints**: 58 503 632

Problem 9.49 (Sharygin 2013). The incircle of triangle ABC touches $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$ at points A', B', and C' respectively. The perpendicular from the incenter I to the C-median meets the line $A'B'$ in point K. Prove that $\overline{CK} \parallel \overline{AB}$. **Hint**: 55 **Sol:** p.277

Problem 9.50 (Shortlist 2004/G2). Let Γ be a circle and let d be a line such that Γ and d have no common points. Further, let $\overline{AB}$ be a diameter of the circle Γ; assume that this diameter $\overline{AB}$ is perpendicular to the line d, and the point B is nearer to the line d than the point A. Let C be an arbitrary point on the circle Γ, different from the points A and B. Let D be the point of intersection of the lines AC and d. One of the two tangents from the point D to the circle Γ touches this circle Γ at a point E; hereby, we assume that the points B and E lie in the same half-plane with respect to the line AC. Denote by F the point of intersection of the lines BE and d. Let the line AF intersect the circle Γ at a point G, different from A.

Prove that the reflection of the point G in the line AB lies on the line CF. **Hints**: 25 285 406 497 **Sol:** p.277

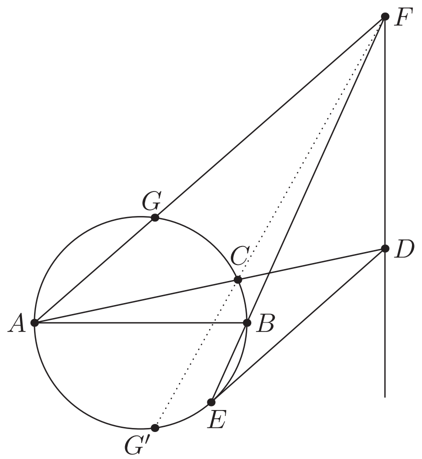

Figure 9.9A. Problem 9.50 is a mouthful.

Problem 9.51 (USA January TST for IMO 2013). Let ABC be an acute triangle. Circle ω_1, with diameter $\overline{AC}$, intersects side $\overline{BC}$ at F (other than C). Circle ω_2, with diameter $\overline{BC}$, intersects side $\overline{AC}$ at E (other than C). Ray AF intersects ω_2 at K and M with $AK < AM$. Ray BE intersects ω_1 at L and N with $BL < BN$. Prove that lines AB, ML, NK are concurrent. **Hints**: 168 374 239

Problem 9.52 (Brazilian Olympiad 2011/5). Let ABC be an acute triangle with orthocenter H and altitudes $\overline{BD}$, $\overline{CE}$. The circumcircle of ADE cuts the circumcircle of ABC at $F \neq A$. Prove that the angle bisectors of $\angle BFC$ and $\angle BHC$ concur at a point on $\overline{BC}$. **Hints**: 405 221 366

Problem 9.53 (ELMO Shortlist 2013). In $\triangle ABC$, a point D lies on line BC. The circumcircle of ABD meets AC at F (other than A), and the circumcircle of ADC meets AB at E (other than A). Prove that as D varies, the circumcircle of AEF always passes through a fixed point other than A, and that this point lies on the median from A to $\overline{BC}$. **Hints**: 511 34 270

Problem 9.54 (APMO 2008/3). Let Γ be the circumcircle of a triangle ABC. A circle passing through points A and C meets the sides $\overline{BC}$ and $\overline{BA}$ at D and E, respectively. The lines AD and CE meet Γ again at G and H, respectively. The tangent lines to Γ at A and C meet the line DE at L and M, respectively.

Prove that the lines LH and MG meet at Γ. **Hints**: 156 444 352 572 **Sol:** p.277

Theorem 9.55 (Brianchon's Theorem). *Let $ABCDEF$ be a hexagon circumscribed about a circle ω. Prove that $\overline{AD}$, $\overline{BE}$, $\overline{CF}$ are concurrent.* **Hints**: 241 35

Problem 9.56 (ELMO Shortlist 2014). Suppose $ABCD$ is a cyclic quadrilateral inscribed in the circle ω. Let $E = \overline{AB} \cap \overline{CD}$ and $F = \overline{AD} \cap \overline{BC}$. Let ω_1, ω_2 be the circumcircles of triangles AEF, CEF, respectively. Let G and H be the intersections of ω and ω_1, ω and ω_2, respectively, with $G \neq A$ and $H \neq C$. Show that $\overline{AC}$, $\overline{BD}$, and $\overline{GH}$ are concurrent. **Hints**: 404 590 443 **Sol:** p.278

Problem 9.57 (ELMO Shortlist 2014). Let $ABCD$ be a cyclic quadrilateral inscribed in circle ω. The tangent to ω at A intersects lines CD and BC at E and F. Lines BE and DF meet ω again G and I, and $H = \overline{BE} \cap \overline{AD}$, $J = \overline{DF} \cap \overline{AB}$. Prove that $\overline{GI}$, $\overline{HJ}$, and the B-symmedian of $\triangle ABC$ are concurrent. **Hints**: 667 234

Problem 9.58 (Shortlist 2005/G6). Let ABC be a triangle, and M the midpoint of its side BC. Let γ be the incircle of triangle ABC. The median AM of triangle ABC intersects the incircle γ at two points K and L. Let the lines passing through K and L, parallel to BC, intersect the incircle γ again in two points X and Y. Let the lines AX and AY intersect BC again at the points P and Q. Prove that $BP = CQ$. **Hints**: 682 543 328 104 563

CHAPTER 10

Complete Quadrilaterals

> Geometry is the art of correct reasoning from incorrectly drawn figures.
>
> Henri Poincaré

This chapter relies on both inversive and projective geometry (Chapters 8 and 9). We study complete quadrilaterals, a frequently recurring configuration in olympiad geometry.

A **complete quadrilateral** consists of four lines, no three concurrent and no two parallel, as well as the six points of intersection they determine. Any quadrilateral (possibly non-convex) with non-parallel sides gives rise to a complete quadrilateral by just extending its sides, and so throughout this chapter we refer to a complete quadrilateral $ABCD$ with $P = \overline{AD} \cap \overline{BC}$ and $Q = \overline{AB} \cap \overline{CD}$,* as in Figure 10.0A.

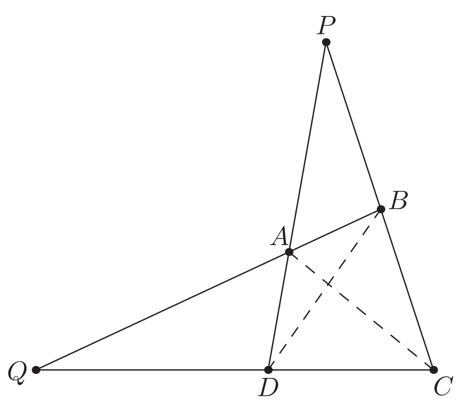

Figure 10.0A. A complete quadrilateral.

This should be reminiscent of Lemma 9.11 and Brocard's theorem (Theorem 9.25). Indeed, the special case where $ABCD$ is cyclic is discussed in Section 10.5.

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* Recall from the Chapter 9, page 170, that $\overline{AB} \cap \overline{XY}$ is shorthand for the intersection of lines AB and XY.

10.1 Spiral Similarity

Before proceeding, we first need to discuss the concept of a spiral similarity. A **spiral similarity** with a center O combines a rotation about O with a dilation. Figure 10.1A gives an example of a spiral similarity.

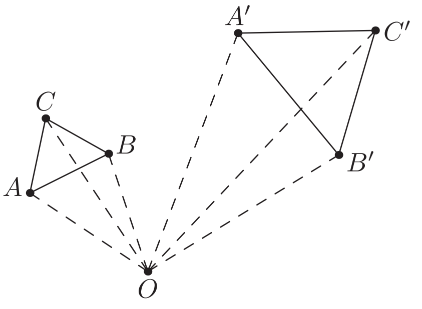


Figure 10.1A. A spiral similarity taking $\triangle ABC$ to $\triangle A'B'C'$.

The most commonly occurring case of a spiral similarity is between two segments. Consider a spiral similarity at O mapping a segment $\overline{AB}$ to $\overline{CD}$, as in Figure 10.1B.

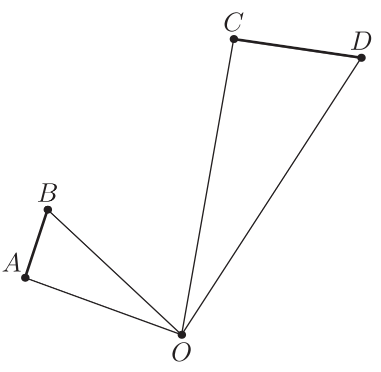


Figure 10.1B. A spiral similarity taking $\overline{AB}$ to $\overline{CD}$.

Of course, $\triangle OAB$ is similar to $\triangle OCD$.

We now determine O in terms of A, B, C, D via complex numbers. It is easy to check that

$$\frac{c-o}{a-o} = \frac{d-o}{b-o}.$$

That implies

$$o = \frac{ad-bc}{a+d-b-c}.$$

So O is uniquely determined by A, B, C, D. That implies in general there is exactly one spiral similarity taking any segment to any other segment. The exception is if $ABDC$ is a parallelogram, since then $a+d=b+c$ and the spiral similarity fails to exist.

This is all fine and well, but where do spiral similarities arise in nature? In fact, they are actually hidden whenever two circles intersect.

Lemma 10.1 (Spiral Centers). *Let $\overline{AB}$ and $\overline{CD}$ be segments, and suppose $X = \overline{AC} \cap \overline{BD}$. If (ABX) and (CDX) intersect again at O, then O is the center of the unique spiral similarity taking $\overline{AB}$ into $\overline{CD}$.*

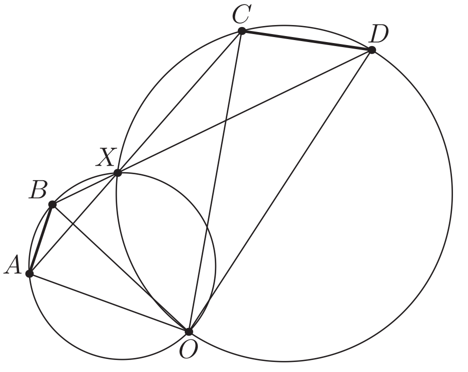


Figure 10.1C. O is the spiral center.

We say "the spiral similarity" instead of "a spiral similarity", because we know already that it is unique.

Proof. This is actually just a matter of angle chasing. We have

$$\measuredangle OAB = \measuredangle OXB = \measuredangle OXD = \measuredangle OCD$$

and similarly

$$\measuredangle OBA = \measuredangle ODC.$$

That implies $\triangle OAB \sim \triangle OCD$, which is sufficient. $\square$

Do not forget this configuration! Whenever all six points in Figure 10.1C appear, we automatically have a pair of similar triangles.

By now, an observant reader may have realized that there is more than one set of similar triangles in Figure 10.1C. We see that in fact, $\triangle OAC \sim \triangle OBD$ as well. After all, $\angle AOC = \angle BOD$ and $\frac{AO}{CO} = \frac{BO}{DO}$ (the ratios arising from the original spiral similarity).

What this means is that spiral similarities occur in pairs. More precisely, we get the following proposition.

Proposition 10.2. *The center of the spiral similarity taking $\overline{AB}$ to $\overline{CD}$ is also the center of the spiral similarity taking $\overline{AC}$ to $\overline{BD}$.*

Thus we have a second spiral similarity, but this time we know its center. What happens if Lemma 10.1 is applied again, this time in the other direction? Does this really mean that $\overline{AB} \cap \overline{CD}$ lies on (AOC) and (BOD) as well? Oh, yes. That is precisely **Miquel's theorem**, discussed in the next section.

10.2 Miquel's Theorem

With these results, we return to our complete quadrilateral $ABCD$ with $P = \overline{AD} \cap \overline{BC}$ and $Q = \overline{AB} \cap \overline{CD}$. We now state one of the most basic results on complete quadrilaterals, namely Miquel's theorem. It is really just the re-interpretation of the spiral similarity in a more natural setting.

Theorem 10.3 (Miquel's Theorem). *The four circles (PAB), (PDC), (QAD), (QBC) concur at the Miquel point M. Furthermore, M is the center of the spiral similarity sending $\overline{AB}$ to $\overline{DC}$ and $\overline{BC}$ to $\overline{AD}$. (In particular, $\triangle MAB \sim \triangle MDC$ and $\triangle MBC \sim \triangle MAD$.)*

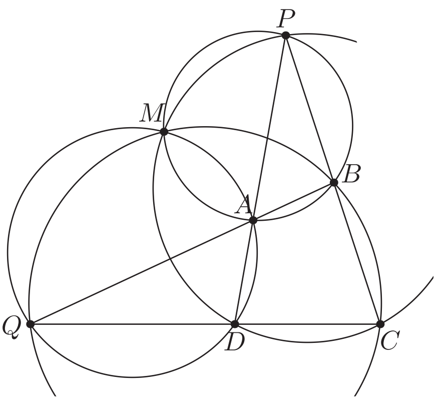


Figure 10.2A. The Miquel point M of a complete quadrilateral.

The point M is called the **Miquel point** of $ABCD$. This is the same Miquel point as in Lemma 1.27; consider triangle PCD with Q, A, B on its sides.

Proof. Define M to be the second intersection of (PAB) and (PDC). By Lemma 10.1, M is the center of the spiral similarity taking $\overline{AB}$ to $\overline{DC}$. Hence, it is also the center of the spiral similarity taking $\overline{BC}$ to $\overline{DA}$. Invoking Lemma 10.1 again, this time in the reverse direction, we see that M lies on (QBC) and (QAD). $\square$

What this means is that *spiral similarity and complete quadrilaterals go hand in hand*. Each gives rise to the other. This gives a powerful way to relate similarities, circles, and intersections to one another.

Problem for this Section

Problem 10.4. Prove that the four circles in Theorem 10.3 concur without appealing to Lemma 10.1. (This is just angle chasing.)

10.3 The Gauss-Bodenmiller Theorem

Consider the three diagonals of a complete quadrilateral, namely $\overline{AC}$, $\overline{BD}$, $\overline{PQ}$. It turns out their midpoints are collinear. The line through them is called the **Gauss line** (sometimes also called the Newton-Gauss line).

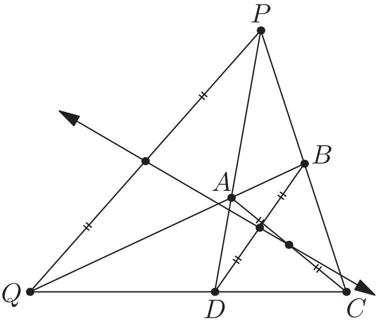


Figure 10.3A. The Gauss line.

Actually, this is a simple corollary of an even more general theorem. Recall that three circles are coaxial if each pair has the same radical axis (see Section 2.4).

Theorem 10.5 (Gauss-Bodenmiller Theorem). *The circles with diameters $\overline{AC}$, $\overline{BD}$, $\overline{PQ}$ are coaxial. Their radical axis is a line passing through each of the four orthocenters of the triangles PAB, PCD, QAD, QBC.*

The radical axis is sometimes called the **Steiner line** (or sometimes **Aubert line**). The figure is shown in Figure 10.3B.

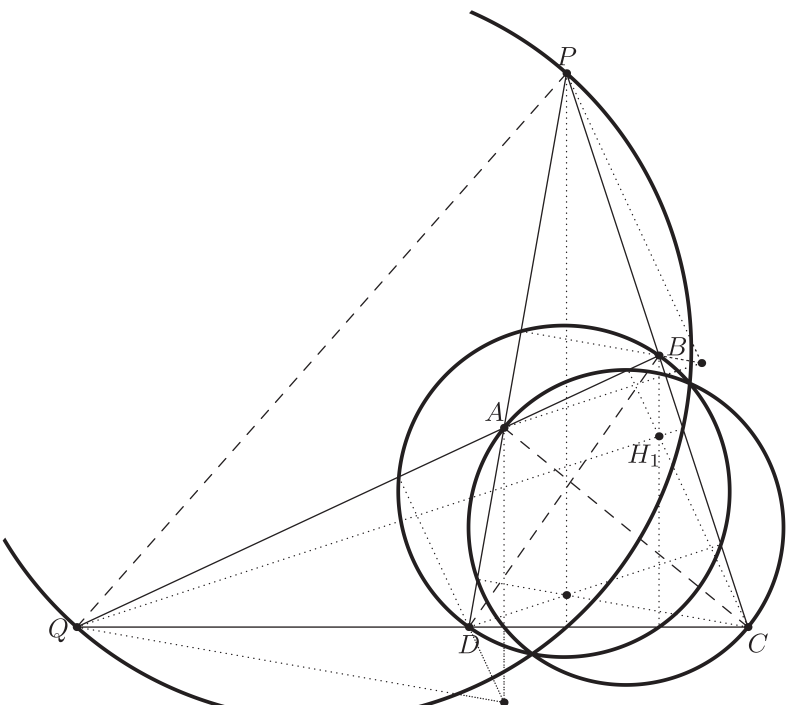


Figure 10.3B. The full form of the Gauss-Bodenmiller theorem.

The proof is surprisingly simple. The idea is to take any orthocenter and show that it has the same power with respect to all three circles. Hence all four orthocenters lie on all the radical axes. This implies the conclusion.

Proof. Let $\omega_1, \omega_2, \omega_3$ denote the circles with diameters $\overline{PQ}$, $\overline{AC}$, $\overline{BD}$, respectively.

Let H_1 denote the orthocenter of triangle BCQ. Check that it is the radical center ω_1, ω_2, and the circle with diameter $\overline{QC}$ (Theorem 2.9). That implies that H_1 lies on the radical axis of the circles ω_1 and ω_2. Doing similar work, we see that H_1 lies on the radical axes of ω_1 and ω_2, ω_2 and ω_3, ω_3 and ω_1.

Similarly, the orthocenters of the other three triangles each lie on all three radical axes. This is only possible if the radical axes of ω_1 and ω_2, ω_2 and ω_3, ω_3 and ω_1 all coincide, as desired. Thus all four orthocenters lie on the desired Steiner line. In particular, the centers of $\omega_1, \omega_2, \omega_3$ all lie on the prescribed Gauss line; this is the line perpendicular to the Steiner line through the centers. □

10.4 More Properties of General Miquel Points

Just for fun, we present two more interesting properties of Miquel points. First, we look more closely at the circles in Miquel's theorem.

Lemma 10.6 (Centers are Concyclic with the Miquel Point). *The four centers of (PAB), (PDC), (QAD), (QBC) lie on a circle passing through the Miquel point.*

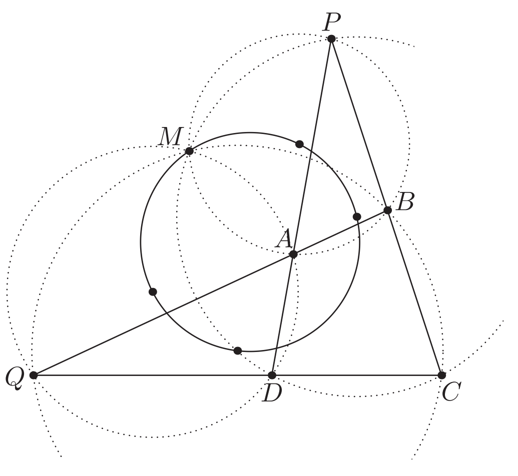


Figure 10.4A. Concyclic centers.

Problem 10.7. If O_1 is the center of (PAB) and O_2 is the center of (PDC), show that $\triangle MO_1O_2 \sim \triangle MAD$. **Hints**: 487 580

Problem 10.8. Establish the main result. **Hint**: 489

Here is one other fun fact. What happens when we drop the perpendiculars from M onto the sides of a complete quadrilateral?

Lemma 10.9 (Altitudes from the Miquel Point). *The feet of the perpendiculars from M to lines AB, BC, CD, DA are collinear. Furthermore, the line though these four points is perpendicular to the Gauss line.*

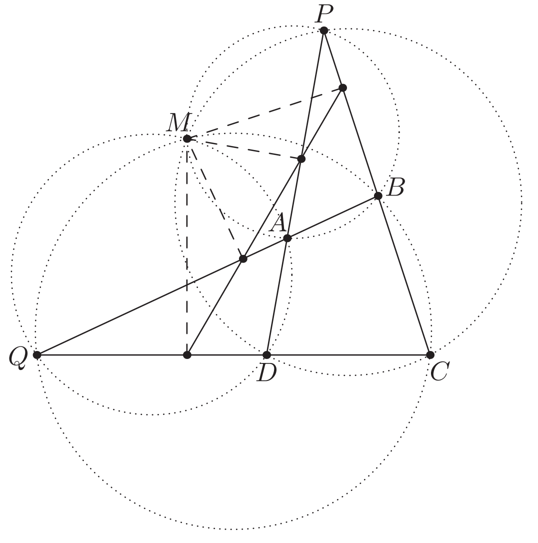


Figure 10.4B. The feet of the altitudes from M are collinear.

Problem 10.10. Prove that the four points are indeed collinear. **Hints**: 385 681

Problem 10.11. Prove that this line is perpendicular to the Gauss line. **Hints**: 90 412 519

10.5 Miquel Points of Cyclic Quadrilaterals

One of the most powerful configurations in olympiad geometry is the Miquel point when complete quadrilateral $ABCD$ is cyclic. In that case, the Miquel point gains several additional properties. All are shadows of the following theorem.

Theorem 10.12 (Miquel Point of a Cyclic Quadrilateral). *Let $ABCD$ be a cyclic quadrilateral inscribed in circle ω with diagonals meeting at R. Then the Miquel point of $ABCD$ is the inverse of R with respect to inversion around ω.*

Proof. Let O be the circumcenter of $ABCD$, and let R^* be the image of R. It suffices to show $R^* = M$. Angle chasing (left as an exercise) lets us establish $\measuredangle AR^*B = \measuredangle APB$, so that R^* lies on (PAB). Similarly, R^* lies on (PCD), (QBC), and (QDA). Hence R^* is indeed the Miquel point. $\square$

Brocard's theorem, anyone? A simple corollary is that the Miquel point M also lies on $\overline{PQ}$. Moreover, if O is the center of ω, then $\overline{OM} \perp \overline{PQ}$. Inversion gives some additional properties, deferred to the exercises.

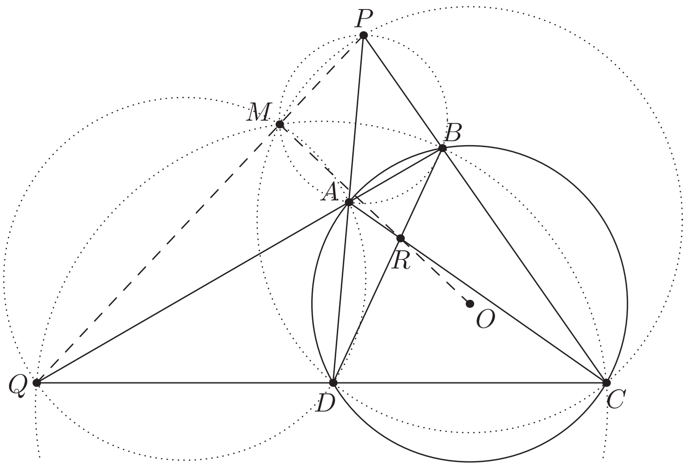


Figure 10.5A. The Miquel point of a cyclic quadrilateral.

Combining these results, we see that the magical Miquel point M has the following properties.

(a) It is the common point of the six circles (OAC), (OBD), (PAD), (PBC), (QAB), (QCD).
(b) It is the center of a spiral similarity taking $\overline{AB}$ to $\overline{CD}$, as well as the spiral similarity taking $\overline{BC}$ to $\overline{DA}$.
(c) It is the inverse of $R = \overline{AC} \cap \overline{BD}$ with respect to an inversion around $(ABCD)$. By Brocard's theorem, M is the foot of O onto $\overline{PQ}$.

Impressive, no? Below we present a few additional properties of the Miquel point M.

Problems for this Section

Problem 10.13. Finish the directed angle chase in the proof of Theorem 10.12. **Hints**: 310 329

Proposition 10.14. *Let M be the Miquel point of cyclic quadrilateral $ABCD$ with circumcenter O. Show that the M is the second intersection of circles (OAC) and (OBD).* **Hint**: 63

Proposition 10.15. *Let M be the Miquel point of cyclic quadrilateral $ABCD$ with circumcenter O. Prove that $\overline{MO}$ bisects $\angle AMC$ and $\angle BMD$.* **Hint**: 398

10.6 Example Problems

To illustrate the results of the Miquel point, we provide as an example a problem appearing on a USA TST for the 54th IMO.

Example 10.16 (USA December TST for IMO 2013). Let ABC be a scalene triangle with $\angle BCA = 90^\circ$, and let D be the foot of the altitude from C. Let X be a point in the interior of the segment CD. Let K be the point on the segment AX such that $BK = BC$.

Similarly, let L be the point on the segment BX such that $AL = AC$. The circumcircle of triangle DKL intersects segment AB at a second point T (other than D).

Prove that $\angle ACT = \angle BCT$.

This is based on the fifth problem from the 2012 IMO, which asked to show that if $\overline{AL}$ and $\overline{BK}$ meet at M, then $ML = MK$.

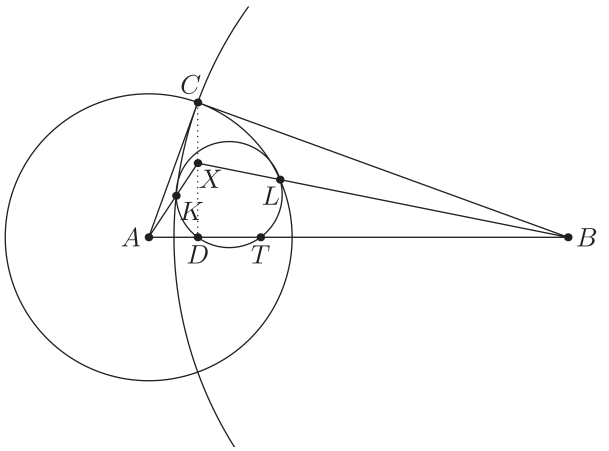


Figure 10.6A. A variation on IMO 2012/5.

The first thing we do is add in the circles ω_A and ω_B centered at A and B passing through C; this lets us cleanly interpret the length condition. Now we have a nice interpretation of the angle condition—the two circles are orthogonal.

Seeing the orthogonal circles, we construct K^* the second intersection of line AK with ω_B. The key observation is that K^* is the image of K under inversion at ω_A, implying that

$$AK \cdot AK^* = AC^2 = AL^2.$$

Similarly, let us construct L^* with $BL \cdot BL^* = BC^2 = BK^2$.

But now something interesting happens. Since X lies on the radical axis of ω_A and ω_B, we find that points K, L, K^*, L^* are concyclic, say on circle ω. Now the above side relations imply that $\overline{AL}$, $\overline{AL^*}$, $\overline{BK}$, $\overline{BK^*}$ are in fact tangents to ω. At this point, if we let $\overline{AL}$ and $\overline{BK}$ intersect at a point M, then $\overline{ML}$ and $\overline{MK}$ are equal tangents; this remark completes the original IMO problem.

Now how can we handle the cyclic quadrilateral $KLTD$? Here Theorem 10.12 comes into play. We recognize D as the Miquel point of cyclic quadrilateral KLK^*L^*. So the point T is none other than the intersection of $\overline{KL^*}$ and $\overline{LK^*}$. This frees us from having to consider (KLD) at all; we simply view T as the intersection of these two sides, lying on $\overline{AB}$ (which is the polar of X).

We focus on ω now. In projective terms, the quadrilateral KLK^*L^* is harmonic, and A and B are the poles of $\overline{LL^*}$ and $\overline{KK^*}$. Let us see if projection gives us any harmonic bundles. If we use our information about tangents, we find

$$-1 = (K, K^*; L, L^*) \stackrel{L}{=} (S, T; A, B).$$

where $S = \overline{KL} \cap \overline{K^*L^*}$ (this lies on $\overline{AB}$ by Brocard's theorem).

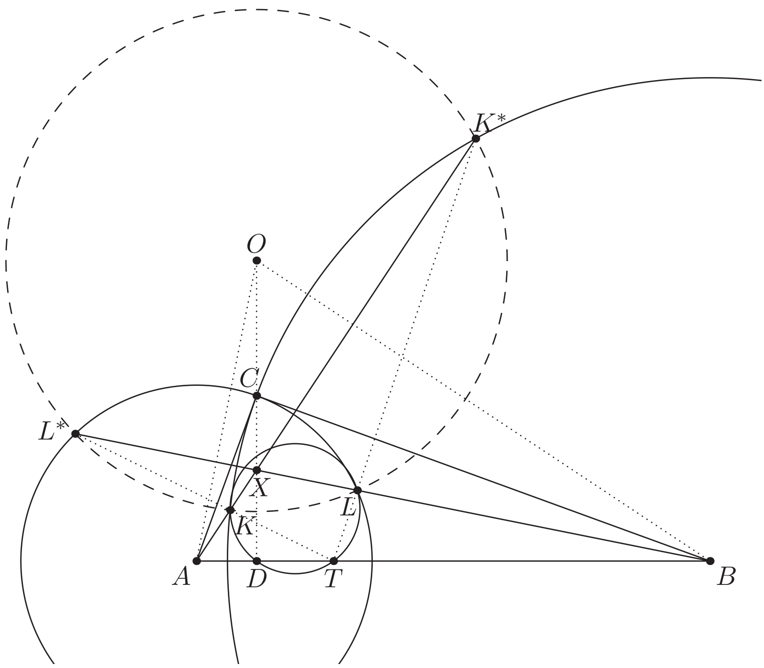


Figure 10.6B. Finding a hidden cyclic quadrilateral.

This is good, since we can apply our Lemma 9.18 now. Unfortunately, this does not finish off the problem. We know that $\angle ACB = 90^\circ$ and $\overline{CA}$ is a bisector of $\angle SCT$, but we actually want $\overline{CT}$ to bisect $\angle ACB$, or equivalently $\angle SCT = 90^\circ$.

The trick now is to consider radical axes. Since triangles XST and XAB are self-polar, by Lemma 9.27 we find that O has the same power with respect to the circles with diameter

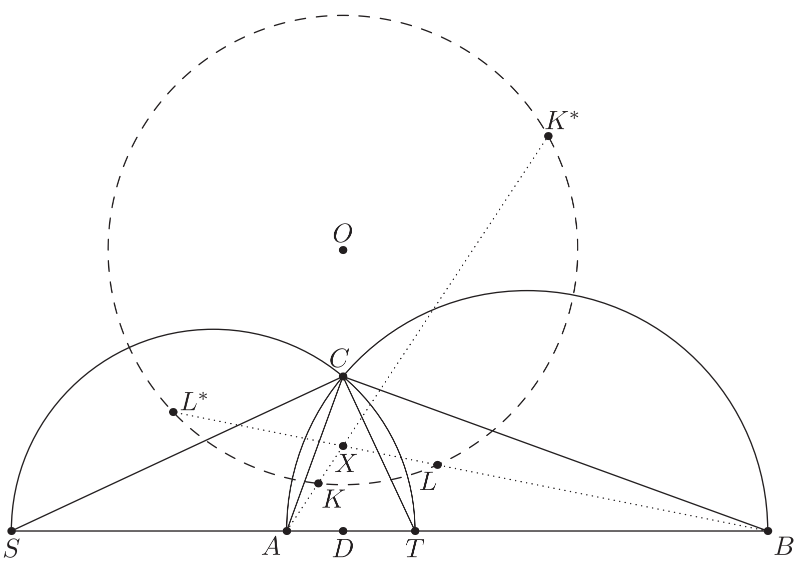


Figure 10.6C. Completing the diagram for Example 10.16.

$\overline{ST}$ and $\overline{AB}$. Hence the radical axis of the circles with diameter $\overline{ST}$ and $\overline{AB}$ contains the point O. Moreover, the radical axis is perpendicular to the line through the centers, namely $\overline{AB}$. This implies it passes through C. Yet C lies on the circle with diameter $\overline{AB}$. Hence it lies on the circle with diameter $\overline{ST}$ as well, as desired.

Solution to Example 10.16. Let ω_A and ω_B be the circles through C centered at A and B; extend rays AK and BL to hit ω_B and ω_A again at K^*, L^*. Evidently KLK^*L^* is cyclic, say with circumcircle ω. Moreover, by orthogonality we observe that $\overline{AL}$, $\overline{AL^*}$, $\overline{BK}$, $\overline{BK^*}$ are tangents to ω (in particular, KLK^*L^* is harmonic).

This means that $\overline{AB}$ is the polar of X. Then D is the Miquel point of cyclic quadrilateral KLK^*L^*, and it follows that $T = \overline{KL^*} \cap \overline{LK^*}$. This implies $-1 = (K, K^*; L, L^*) \stackrel{L}{=} (S, T; A, B)$ where $S = \overline{KL} \cap \overline{K^*L^*}$. Hence it suffices to prove $\angle SCT = 90^\circ$.

As triangles XST and XAB are self-polar to ω, it follows that O has the same power to the circles with diameter $\overline{ST}$ and $\overline{AB}$. Hence the radical axis of these two circles is line OC; this means C lies on the circle with diameter $\overline{ST}$ and we are done. □

10.7 Problems

Problem 10.17 (NIMO 2014). Let ABC be an acute triangle with orthocenter H and let M be the midpoint of $\overline{BC}$. Denote by ω_B the circle passing through B, H, and M, and denote by ω_C the circle passing through C, H, and M. Lines AB and AC meet ω_B and ω_C again at P and Q, respectively. Rays PH and QH meet ω_C and ω_B again at R and S, respectively. Show that $\triangle BRS$ and $\triangle CRS$ have the same area. **Hints**: 268 633 556

Problem 10.18 (USAMO 2013/1). In triangle ABC, points P, Q, R lie on sides BC, CA, AB, respectively. Let ω_A, ω_B, ω_C denote the circumcircles of triangles AQR, BRP, CPQ, respectively. Given the fact that segment AP intersects ω_A, ω_B, ω_C again at X, Y, Z respectively, prove that $YX/XZ = BP/PC$. **Hints**: 59 92 382 686

Problem 10.19 (Shortlist 1995/G8). Suppose that $ABCD$ is a cyclic quadrilateral. Let $E = \overline{AC} \cap \overline{BD}$ and $F = \overline{AB} \cap \overline{CD}$. Prove that F lies on the line joining the orthocenters of triangles EAD and EBC. **Hints**: 428 416 **Sol:** p.278

Problem 10.20 (USA TST 2007/1). Circles ω_1 and ω_2 meet at P and Q. Segments AC and BD are chords of ω_1 and ω_2 respectively, such that segment AB and ray CD meet at P. Ray BD and segment AC meet at X. Point Y lies on ω_1 such that $\overline{PY} \parallel \overline{BD}$. Point Z lies on ω_2 such that $\overline{PZ} \parallel \overline{AC}$. Prove that points Q, X, Y, Z are collinear. **Hints**: 277 615 525 **Sol:** p.279

Problem 10.21 (USAMO 2013/6). Let ABC be a triangle. Find all points P on segment BC satisfying the following property: If X and Y are the intersections of line PA with the common external tangent lines of the circumcircles of triangles PAB and PAC, then

$$\left(\frac{PA}{XY}\right)^2 + \frac{PB \cdot PC}{AB \cdot AC} = 1.$$

Hints: 196 68 42 327

Problem 10.22 (USA TST 2007/5). Acute triangle ABC is inscribed in circle ω. The tangent lines to ω at B and C meet at T. Point S lies on ray BC such that $\overline{AS} \perp \overline{AT}$. Points

B_1 and C_1 lie on ray ST (with C_1 in between B_1 and S) such that $B_1T = BT = C_1T$. Prove that triangles ABC and AB_1C_1 are similar. **Hints**: 199 375 293 377 **Sol:** p.280

Problem 10.23 (IMO 2005/2). Let $ABCD$ be a fixed convex quadrilateral with $BC = DA$ and $\overline{BC} \nparallel \overline{DA}$. Let two variable points E and F lie of the sides BC and DA, respectively, and satisfy $BE = DF$. The lines AC and BD meet at P, the lines BD and EF meet at Q, the lines EF and AC meet at R. Prove that the circumcircles of the triangles PQR, as E and F vary, have a common point other than P. **Hints**: 562 436 481 499 **Sol:** p.280

Problem 10.24 (USAMO 2006/6). Let $ABCD$ be a quadrilateral, and let E and F be points on sides AD and BC, respectively, such that $\frac{AE}{ED} = \frac{BF}{FC}$. Ray FE meets rays BA and CD at S and T, respectively. Prove that the circumcircles of triangles SAE, SBF, TCF, and TDE pass through a common point. **Hints**: 617 319 493

Problem 10.25 (Balkan Olympiad 2009/2). Let $\overline{MN}$ be a line parallel to the side BC of a triangle ABC, with M on the side AB and N on the side AC. The lines $\overline{BN}$ and $\overline{CM}$ meet at point P. The circumcircles of triangles BMP and CNP intersect at a point $Q \neq P$. Prove that $\angle BAQ = \angle CAP$. **Hints**: 636 358 208 399

Problem 10.26 (USA TSTST 2012/7). Triangle ABC is inscribed in circle Ω. The interior angle bisector of angle A intersects side BC and Ω at D and L (other than A), respectively. Let M be the midpoint of $\overline{BC}$. The circumcircle of triangle ADM intersects sides AB and AC again at Q and P (other than A), respectively. Let N be the midpoint of $\overline{PQ}$, and let H be the foot of the perpendicular from L to line ND.

Prove that line ML is tangent to the circumcircle of triangle HMN. **Hints**: 494 517 193 604 **Sol:** p.281

Problem 10.27 (USA TSTST 2012/2). Let $ABCD$ be a quadrilateral with $AC = BD$. Diagonals AC and BD meet at P. Let ω_1 and O_1 denote the circumcircle and the circumcenter of triangle ABP. Let ω_2 and O_2 denote the circumcircle and circumcenter of triangle CDP. Segment BC meets ω_1 and ω_2 again at S and T (other than B and C), respectively. Let M and N be the midpoints of minor arcs $\widehat{SP}$ (not including B) and $\widehat{TP}$ (not including C). Prove that $\overline{MN} \parallel \overline{O_1O_2}$. **Hints**: 81 261 312

Problem 10.28 (USA TST 2009/2). Let ABC be an acute triangle. Point D lies on side BC. Let O_B, O_C be the circumcenters of triangles ABD and ACD, respectively. Suppose that the points B, C, O_B, O_C lie on a circle centered at X. Let H be the orthocenter of triangle ABC. Prove that $\angle DAX = \angle DAH$. **Hints**: 95 163

Problem 10.29 (Shortlist 2009/G4). Given a cyclic quadrilateral $ABCD$, let the diagonals AC and BD meet at E and the lines AD and BC meet at F. The midpoints of $\overline{AB}$ and $\overline{CD}$ are G and H, respectively. Show that $\overline{EF}$ is tangent at E to the circle through the points E, G, and H. **Hints**: 222 56 413 627 **Sol:** p.281

Problem 10.30 (Shortlist 2006/G9). Points A_1, B_1, C_1 are chosen on the sides BC, CA, AB of a triangle ABC respectively. The circumcircles of triangles AB_1C_1, BC_1A_1, CA_1B_1 intersect the circumcircle of triangle ABC again at points A_2, B_2, C_2 respectively ($A_2 \neq A$, $B_2 \neq B$, $C_2 \neq C$). Points A_3, B_3, C_3 are symmetric to A_1, B_1, C_1 with respect to the midpoints of the sides BC, CA, AB respectively. Prove that the triangles $A_2B_2C_2$ and $A_3B_3C_3$ are similar. **Hints**: 10 606 680 14 **Sol:** p.282

Problem 10.31 (Shortlist 2005/G5). Let $\triangle ABC$ be an acute-angled triangle with $AB \neq AC$. Let H be the orthocenter of triangle ABC, and let M be the midpoint of the side BC. Let D be a point on the side AB and E a point on the side AC such that $AE = AD$ and the points D, H, E are on the same line. Prove that the line HM is perpendicular to the common chord of the circumcircles of $\triangle ABC$ and $\triangle ADE$. **Hints**: 585 254 99 625 640 98 53 250

CHAPTER 11

Personal Favorites

> Graders received some elegant solutions, some not-so-elegant solutions, and some so-not-elegant solutions.
> MOP 2012

Here are some fairly nice problems taken from various sources. Full solutions to all problems can be found in Appendix C.4.

Problem 11.0. Find as many typos in this book as you can.

Problem 11.1 (Canada 2000/4). Let $ABCD$ be a convex quadrilateral with $\angle CBD = 2\angle ADB$, $\angle ABD = 2\angle CDB$ and $AB = CB$. Prove that $AD = CD$. **Hints**: 573 534 612

Problem 11.2 (EGMO 2012/1). Let ABC be a triangle with circumcenter O. The points D, E, F lie in the interiors of the sides BC, CA, AB respectively, such that $\overline{DE}$ is perpendicular to $\overline{CO}$ and $\overline{DF}$ is perpendicular to $\overline{BO}$. Let K be the circumcenter of triangle AFE. Prove that the lines $\overline{DK}$ and $\overline{BC}$ are perpendicular. **Hints**: 305 541

Problem 11.3 (ELMO 2013/4). Triangle ABC is inscribed in circle ω. A circle with chord BC intersects segments AB and AC again at S and R, respectively. Segments BR and CS meet at L, and rays LR and LS intersect ω at D and E, respectively. The internal angle bisector of $\angle BDE$ meets line ER at K. Prove that if $BE = BR$, then $\angle ELK = \frac{1}{2}\angle BCD$. **Hints**: 213 568 44 538

Problem 11.4 (Sharygin 2012). Let $\overline{BM}$ be the median of right-angled triangle ABC with $\angle B = 90^\circ$. The incircle of triangle ABM touches sides AB and AM in points A_1 and A_2; points C_1, C_2 are defined similarly. Prove that lines A_1A_2 and C_1C_2 meet on the bisector of angle ABC. **Hints**: 658 340

Problem 11.5 (USAMTS). In quadrilateral $ABCD$, $\angle DAB = \angle ABC = 110^\circ$, $\angle BCD = 35^\circ$, $\angle CDA = 105^\circ$, and $\overline{AC}$ bisects $\angle DAB$. Find $\angle ABD$. **Hints**: 559 397 423 259

Problem 11.6 (MOP 2012). Let ABC be an acute triangle with circumcenter ω and altitudes $\overline{AD}$, $\overline{BE}$, $\overline{CF}$. Circle γ is the image of ω when reflected across $\overline{AB}$. Ray EF meets ω at P, and ray DF meets γ at Q. Prove that the points B, P, Q are collinear. **Hints**: 262 679 337 694

Problem 11.7 (Sharygin 2013). Chords $\overline{BC}$ and $\overline{DE}$ of circle ω meet at point A. The line through D parallel to $\overline{BC}$ meets ω again at F, and $\overline{FA}$ meets ω again at T. Let M denote the intersection of $\overline{ET}$ and $\overline{BC}$, and let N be the reflection of A over M. Show that the circumcircle of $\triangle DEN$ passes through the midpoint of $\overline{BC}$. **Hints**: 600 127 209 37

Problem 11.8 (ELMO 2012/1). In acute triangle ABC, let D, E, F denote the feet of the altitudes from A, B, C, respectively, and let ω be the circumcircle of $\triangle AEF$. Let ω_1 and ω_2 be the circles through D tangent to ω at E and F, respectively. Show that ω_1 and ω_2 meet at a point P on line BC other than D. **Hints**: 289 131 298 510

Problem 11.9 (Sharygin 2013). In trapezoid $ABCD$, $\angle A = \angle D = 90°$. Let M and N be the midpoints of diagonals AC and BD, respectively. Line BC meets (ABN) and (CDM) again at Q and R. If K is the midpoint of $\overline{MN}$, show that $KQ = KR$. **Hints**: 669 232 146

Problem 11.10 (Bulgarian Olympiad 2012). Let ABC be a triangle with circumcircle Ω and let P be a variable point in its interior. The rays PA, PB, PC meet Ω again at A_1, B_1, C_1. Let A_2 denote the reflection of A_1 over $\overline{BC}$, and define B_2 and C_2 similarly. Prove that the circumcircle of triangle $A_2B_2C_2$ passes through a fixed point independent of P. **Hints**: 464 427 430 311 631

Problem 11.11 (Sharygin 2013). Points A_1, B_1, C_1, A_2, B_2, C_2 lie inside a triangle ABC so that A_1 is on $\overline{AB_1}$, B_1 is on $\overline{BC_1}$, C_1 is on $\overline{CA_1}$, A_2 is on $\overline{AC_2}$, B_2 is on $\overline{BA_2}$, C_2 is on $\overline{CB_2}$. Suppose the angles BAA_1, CBB_1, ACC_1, CAA_2, ABB_2, BCC_2 are equal. Prove that $\triangle A_1B_1C_1$ and $\triangle A_2B_2C_2$ are congruent. **Hints**: 388 637 485 88

Problem 11.12 (Sharygin 2013). Let ABC be a triangle, and let $\overline{AD}$ denote the bisector of $\angle A$ (with D on $\overline{BC}$). Points M and N are the projections of B and C respectively to $\overline{AD}$. The circle with diameter $\overline{MN}$ intersects $\overline{BC}$ at points X and Y.

Prove that $\angle BAX = \angle CAY$. **Hints**: 300 75 471 583

Problem 11.13 (USA December TST for IMO 2015). Let ABC be a scalene triangle with incenter I whose incircle is tangent to $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ at D, E, F, respectively. Denote by M the midpoint of $\overline{BC}$ and let P be a point in the interior of $\triangle ABC$ so that $MD = MP$ and $\angle PAB = \angle PAC$. Let Q be a point on the incircle such that $\angle AQD = 90°$. Prove that either $\angle PQE = 90°$ or $\angle PQF = 90°$. **Hints**: 415 263 368 504

Problem 11.14 (EGMO 2014/2). Let D and E be points in the interiors of sides AB and AC, respectively, of triangle ABC, such that $DB = BC = CE$. Lines CD and BE meet at F. Prove that the incenter I of triangle ABC, the orthocenter H of triangle DEF, and the midpoint M of arc BAC of the circumcircle of triangle ABC are collinear. **Hints**: 392 108 692 512 630

Problem 11.15 (Online Math Open Winter 2013). In $\triangle ABC$, $CA = 1960\sqrt{2}$, $CB = 6720$, and $\angle C = 45°$. Let K, L, M lie on lines BC, CA, and AB such that $\overline{AK} \perp \overline{BC}$, $\overline{BL} \perp \overline{CA}$, and $AM = BM$. Let N, O, P lie on $\overline{KL}$, $\overline{BA}$, and $\overline{BL}$ such that $AN = KN$, $BO = CO$, and A lies on line NP.

If H is the orthocenter of $\triangle MOP$, compute HK^2. **Hints**: 629 527 33 433 516 330 105

Problem 11.16 (USAMO 2007/6). Let ABC be an acute triangle with ω, S, and R being its incircle, circumcircle, and circumradius, respectively. Circle ω_A is tangent internally to S at A and tangent externally to ω. Circle S_A is tangent internally to S at A and tangent internally to ω.

Let P_A and Q_A denote the centers of ω_A and S_A, respectively. Define points P_B, Q_B, P_C, Q_C analogously. Prove that

$$8P_AQ_A \cdot P_BQ_B \cdot P_CQ_C \le R^3$$

with equality if and only if triangle ABC is equilateral. **Hints**: 292 391 235

Problem 11.17 (Sharygin 2013). Let ABC be a triangle with angle bisector $\overline{AL}$ (where L is on $\overline{BC}$). Points O_1 and O_2 are the circumcenters of $\triangle ABL$ and $\triangle ACL$ respectively, and points B_1 and C_1 are the projections of C and B to the bisectors of angles B and C respectively. The incircle of a triangle ABC touches $\overline{AC}$ and $\overline{AB}$ at points B_0 and C_0 respectively, and the bisectors of angles B and C meet the perpendicular bisector of $\overline{AL}$ at points Q and P respectively.

Prove that the five lines $\overline{PC_0}$, $\overline{QB_0}$, $\overline{O_1C_1}$, $\overline{O_2B_1}$ and $\overline{BC}$ are all concurrent. **Hints**: 331 484 158 142

Problem 11.18 (January TST for IMO 2015). Let ABC be a non-equilateral triangle and let M_A, M_B, M_C be the midpoints of the sides BC, CA, AB, respectively. Let S be a point lying on the Euler line. Denote by X, Y, Z the second intersections of $\overline{M_AS}$, $\overline{M_BS}$, $\overline{M_CS}$ with the nine-point circle. Prove that $\overline{AX}$, $\overline{BY}$, $\overline{CZ}$ are concurrent. **Hints**: 176 182 369 546

Problem 11.19 (Iran TST 2009/9). Let ABC be a triangle with incenter I and contact triangle DEF. Let M be the foot of the perpendicular from D to $\overline{EF}$ and let P be the midpoint of $\overline{DM}$. If H is the orthocenter of triangle BIC, prove that $\overline{PH}$ bisects $\overline{EF}$. **Hints**: 223 288 434 269 609 215 505 438

Problem 11.20 (IMO 2011/6). Let ABC be an acute triangle with circumcircle Γ. Let ℓ be a tangent line to Γ, and let ℓ_a, ℓ_b, ℓ_c be the lines obtained by reflecting ℓ in the lines BC, CA, and AB, respectively. Show that the circumcircle of the triangle determined by the lines ℓ_a, ℓ_b, and ℓ_c is tangent to the circle Γ. **Hints**: 685 227 39 387 363 113 531

Problem 11.21 (Taiwan TST 2014). Let ABC be a triangle with circumcircle Γ and let M be an arbitrary point on Γ. Suppose the tangents from M to the incircle of ABC intersect $\overline{BC}$ at two distinct points X_1 and X_2. Prove that the circumcircle of triangle MX_1X_2 passes through the tangency point of the A-mixtilinear incircle with Γ. **Hints**: 422 306 498 566 389 624

Problem 11.22 (Taiwan TST 2015). In scalene triangle ABC with incenter I, the incircle is tangent to sides CA and AB at points E and F. The tangents to the circumcircle of $\triangle AEF$ at E and F meet at S. Lines EF and BC intersect at T. Prove that the circle with diameter $\overline{ST}$ is orthogonal to the nine-point circle of $\triangle BIC$. **Hints**: 150 189 507 582 135 264

Part IV
Appendices

APPENDIX **A**

An Ounce of Linear Algebra

Many of the computational techniques invoke properties of determinants and vectors. We describe in detail the relevant parts of the technology here.

A.1 Matrices and Determinants

A **matrix** (plural **matrices**) is a rectangular array of numbers, for example

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

Throughout this text, we will be mostly concerned with 2×2 and 3×3 matrices.

A **determinant** of a matrix A, denoted $\det A$ or $|A|$, is a special value associated with the matrix A. (When the matrix is written in full, we replace the brackets with bars.) Determinants feature prominently in Chapter 7 and also in Chapters 5 and 6.

We define only the determinant of a 2×2 matrix and a 3×3 matrix. We have

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

for a 2×2 matrix. For a 3×3 matrix we have

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} + b_1 \begin{vmatrix} c_2 & c_3 \\ a_2 & a_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}$$

or equivalently

$$a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} + a_2 \begin{vmatrix} b_3 & b_1 \\ c_3 & c_1 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}.$$

In the definition, the 2×2 sub-matrices are called **minors**.

Determinants are nice because there are clean ways to evaluate them. For example, we have the following properties, which we state without proof.

Proposition A.1 (Swapping Rows or Columns). *Let A be a matrix, and B be a matrix formed by swapping either a pair of rows or a pair of columns in A. Then* $\det A = -\det B$.

Proposition A.2 (Factoring). *We have*

$$\begin{vmatrix} ka_1 & a_2 & a_3 \\ kb_1 & b_2 & b_3 \\ kc_1 & c_2 & c_3 \end{vmatrix} = k \cdot \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

Similar statements hold for the other rows and columns.

Most surprisingly, we can actually add and subtract rows and columns from each other!

Theorem A.3 (Elementary Row Operations). *For any real number k, we have*

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 + kb_1 & a_2 + kb_2 & a_3 + kb_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

Analogous operations can be performed on the other rows and columns.

In other words, we can add and subtract multiples of rows or columns from each other without affecting the determinant. This often lets us eliminate terms that recur frequently across the determinant.

Here is an example. Suppose we wish to evaluate the determinant

$$\begin{vmatrix} \frac{1}{2}\left(p+a+c-\frac{ac}{p}\right) & \frac{1}{2}\left(\frac{1}{p}+\frac{1}{a}+\frac{1}{c}-\frac{p}{ca}\right) & 1 \\ \frac{1}{2}\left(p+a+b-\frac{ab}{p}\right) & \frac{1}{2}\left(\frac{1}{p}+\frac{1}{a}+\frac{1}{b}-\frac{p}{ba}\right) & 1 \\ \frac{1}{2}(p+a+b+c) & \frac{1}{2}\left(\frac{1}{p}+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) & 1 \end{vmatrix}.$$

Straight multiplication would be rather horrible. Fortunately, we can eliminate a lot of common terms. First, we can pull out all the factors of $\frac{1}{2}$ to get

$$\frac{1}{4}\begin{vmatrix} p+a+c-\frac{ac}{p} & \frac{1}{p}+\frac{1}{a}+\frac{1}{c}-\frac{p}{ca} & 1 \\ p+a+b-\frac{ab}{p} & \frac{1}{p}+\frac{1}{a}+\frac{1}{b}-\frac{p}{ba} & 1 \\ p+a+b+c & \frac{1}{p}+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1 \end{vmatrix}.$$

Now noticing the plethora of common terms, we decide to subtract $p+a+b+c$ times the third column from the first column. This gives

$$\frac{1}{4}\begin{vmatrix} -b-\frac{ac}{p} & \frac{1}{p}+\frac{1}{a}+\frac{1}{c}-\frac{p}{ca} & 1 \\ -c-\frac{ab}{p} & \frac{1}{p}+\frac{1}{a}+\frac{1}{b}-\frac{p}{ba} & 1 \\ 0 & \frac{1}{p}+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1 \end{vmatrix}.$$

Similarly, we can knock out $\frac{1}{p}+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$ times the third column from the first. We obtain

$$\frac{1}{4}\begin{vmatrix} -b-\frac{ac}{p} & -\frac{1}{b}-\frac{p}{ca} & 1 \\ -c-\frac{ab}{p} & -\frac{1}{c}-\frac{p}{ba} & 1 \\ 0 & 0 & 1 \end{vmatrix} = \frac{1}{4}\begin{vmatrix} b+\frac{ac}{p} & \frac{1}{b}+\frac{p}{ca} & 1 \\ c+\frac{ab}{p} & \frac{1}{c}+\frac{p}{ba} & 1 \\ 0 & 0 & 1 \end{vmatrix}.$$

Here we have also taken the liberty of factoring out the two minus signs. Now this determinant looks much tamer, and we can evaluate by minors. Because of the 0s in the last row, we use minors on the last row: we find

$$\frac{1}{4}\left(0\begin{vmatrix}\frac{1}{b}+\frac{p}{ca} & 1\\ \frac{1}{c}+\frac{p}{ba} & 1\end{vmatrix}+0\begin{vmatrix}1 & b+\frac{ac}{p}\\ 1 & c+\frac{ab}{p}\end{vmatrix}+1\begin{vmatrix}b+\frac{ac}{p} & \frac{1}{b}+\frac{p}{ca}\\ c+\frac{ab}{p} & \frac{1}{c}+\frac{p}{ba}\end{vmatrix}\right).$$

Now we have only one determinant to compute! We can just expand it as

$$\frac{1}{4}\left[\left(b+\frac{ac}{p}\right)\left(\frac{1}{c}+\frac{p}{ba}\right)-\left(\frac{1}{b}+\frac{p}{ca}\right)\left(c+\frac{ab}{p}\right)\right].$$

Conveniently enough, this expands to zero! If you have read Chapter 6, then you might realize that this actually establishes Lemma 4.4 using complex numbers. (Why?)

A.2 Cramer's Rule

Cramer's rule is a method for converting a system of equations into a determinant. It also is a good illustration of row and column operations, so we present it below.

Theorem A.4 (Cramer's Rule). *Consider a system of equations*

$$\begin{aligned} a_x x + a_y y + a_z z &= a\\ b_x x + b_y y + b_z z &= b\\ c_x x + c_y y + c_z z &= c. \end{aligned}$$

Then the solution for x is

$$x = \begin{vmatrix} a & a_y & a_z\\ b & b_y & b_z\\ c & c_y & c_z \end{vmatrix} \div \begin{vmatrix} a_x & a_y & a_z\\ b_x & b_y & b_z\\ c_x & c_y & c_z \end{vmatrix}$$

provided the denominator is nonzero. Analogous equations hold for y and z.

Proof. The numerator is

$$\begin{vmatrix} a_x x + a_y y + a_z z & a_y & a_z\\ b_x x + b_y y + b_z z & b_y & b_z\\ c_x x + c_y y + c_z z & c_y & c_z \end{vmatrix} = \begin{vmatrix} a_x x & a_y & a_z\\ b_x x & b_y & b_z\\ c_x x & c_y & c_z \end{vmatrix}.$$

Here we have subtracted y times the second column and z times the third column from the first. Factoring, the numerator equals

$$x\begin{vmatrix} a_x & a_y & a_z\\ b_x & b_y & b_z\\ c_x & c_y & c_z \end{vmatrix}.$$

□

A.3 Vectors and the Dot Product

Vectors provide the most basic notion of addition in the plane, and thus form the foundation for our analytic tools.

In the linear algebra realm, a vector is just an arrow with both a magnitude (length) and a direction. A vector pointing from a point A to a point B is denoted $\overrightarrow{AB}$. In order to

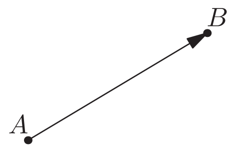


Figure A1. A vector pointing from A to B.

associate points to vectors, we usually define a single point O as the origin, or **zero vector**. Then we associate every point P with the vector $\overrightarrow{OP}$, abbreviated as just $\vec{P}$. This is much like complex numbers; indeed, the two concepts are ofter used interchangeably.

Vectors thus can be represented coordinate-wise: in the plane, the vector pointing to (x, y) in the Cartesian plane (from $(0, 0)$) is denoted $\langle x, y\rangle$. The zero vector is then $\langle 0, 0\rangle$. The magnitude of a vector $\vec{v}$ is written $|\vec{v}|$.

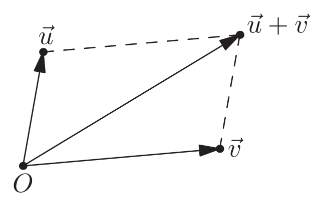


Figure A2. Adding two vectors.

Vectors add exactly as one would expect: the sum of $\langle x_1, y_1\rangle$ and $\langle x_2, y_2\rangle$ is $\langle x_1 + y_1, x_2 + y_2\rangle$. A second interpretation of this addition is the parallelogram law, illustrated in Figure A2.

Vectors can also be scaled by real numbers by simply adjusting their magnitude.

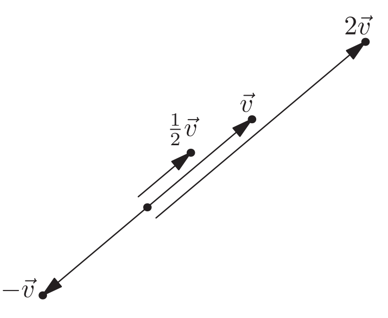


Figure A3. Vectors can also be scaled by constants.

It is important to note that with this scaling, we can take weighted averages of vectors and get the expected results. For example, given segment $\overline{AB}$ with midpoint M, we have $\vec{M} = \frac{1}{2}\left(\vec{A} + \vec{B}\right)$.

Vanilla vectors are not used too often in olympiad problems: rather, we use one of our well-established systems built on top of them (for example, Cartesian coordinates, complex numbers, or barycentric coordinates). However, there is one concept from vectors that can be useful: the dot product.

The **dot product** of two vectors v and w is given by

$$\vec{v} \cdot \vec{w} = |\vec{v}|\,|\vec{w}| \cos\theta$$

where θ is the angle made between the two vectors. Surprisingly, it turns out that

$$\langle a, b\rangle \cdot \langle x, y\rangle = ax + by.$$

The dot product provides a way to multiply vectors, different from the multiplication of complex numbers. It has the following properties:

- The dot product is distributive, commutative, and associative, so you can treat it like multiplication.
- We can express the magnitude of $\vec{v}$ in terms of the dot product by $|\vec{v}|^2 = \vec{v} \cdot \vec{v}$.
- Two (nonzero) vectors $\vec{v}$ and $\vec{w}$ are perpendicular if and only if $\vec{v} \cdot \vec{w} = 0$.

To see an application of this, consider a triangle ABC with circumcenter O. If we set $\vec{O}$ as the zero vector $\vec{0}$, then we have the nice property that

$$\left|\vec{A}\right| = \left|\vec{B}\right| = \left|\vec{C}\right| = R$$

where R is of course the circumradius. So that means $\vec{A} \cdot \vec{A} = R^2$, and so on.

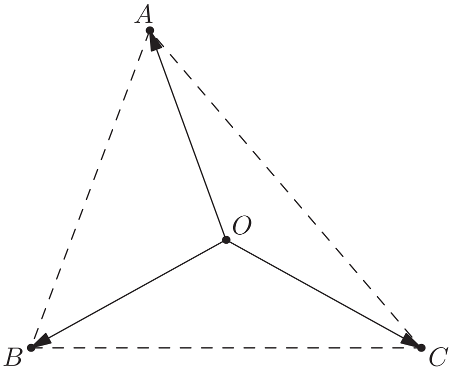


Figure A4. Tossing $\triangle ABC$ into a vector system.

Now what of $\vec{A} \cdot \vec{B}$? By definition, this is $R^2 \cos 2C$. But $\cos 2C = 1 - 2\sin^2 C = 1 - 2\left(\frac{c}{2R}\right)^2$, and accordingly we discover

$$\vec{A} \cdot \vec{B} = R^2 - \frac{1}{2}c^2.$$

Similarly, $\vec{B} \cdot \vec{C} = R^2 - \frac{1}{2}a^2$ and $\vec{C} \cdot \vec{A} = R^2 - \frac{1}{2}b^2$.

Now in Chapter 6 we show that the orthocenter H of ABC is actually given by the simple formula $\vec{H} = \vec{A} + \vec{B} + \vec{C}$. That means, for example, that we can compute OH! It is just a matter of evaluating the dot product.

$$\begin{aligned}
OH^2 &= \left|\overrightarrow{OH}\right|^2 = \left|\vec{H}\right|^2 \\
&= \vec{H} \cdot \vec{H} \\
&= \left(\vec{A} + \vec{B} + \vec{C}\right) \cdot \left(\vec{A} + \vec{B} + \vec{C}\right) \\
&= \vec{A} \cdot \vec{A} + \vec{B} \cdot \vec{B} + \vec{C} \cdot \vec{C} \\
&\quad + 2\left(\vec{A} \cdot \vec{B} + \vec{B} \cdot \vec{C} + \vec{C} \cdot \vec{A}\right) \\
&= 3R^2 + 2\left(3R^2 - \frac{1}{2}\left(a^2 + b^2 + c^2\right)\right) \\
&= 9R^2 - a^2 - b^2 - c^2.
\end{aligned}$$

We will use these properties again to prove theorems in Chapter 7, when we construct a distance formula and a perpendicularity criterion for barycentric coordinates.

APPENDIX B

Hints

1. Try angle chasing; you might see it.
2. Construct circles.
3. The ratio is just $\sqrt{2}$.
4. Something is concurrent. Draw a good diagram.
5. We can compute the angles that $\overline{BJ}$ makes with $\angle B$.
6. It is enough to take $P = (0, s, t)$ with $s + t = 1$ and do some computation.
7. Pick a point to handle the weird angle condition.
8. You cannot take half of directed angles! How you can get around this?
9. Trigonometric form of Ceva's theorem.
10. Spiral similarity, of course, but also length ratios.
11. Find the homothety.
12. Let A_0 be the intersection of lines B_1C_1 and BC.
13. This is very hard for a G1, which is why there was no easy geometry at the IMO 2011.
14. Prove that $\triangle A_2BC \sim \triangle AC_3B_3$.
15. Do you see a pair of perpendicular lines?
16. Look at triangle BPC.
17. Remember Lemma 2.11.
18. Which quadrilateral is cyclic?
19. Use angle chasing to show that $APOQ$ is cyclic, thus we're done.
20. There is a right angle and we want an angle bisector. Which configuration does this remind you of?
21. Directed angles will fail here because the condition that X and A are on different arcs is necessary.
22. Do some computation with the inversive distance formula. The answer should pop right out.
23. Radical axis.
24. First recall that $ME = MF = MB = MC$.
25. This can be solved in a lot of ways, but there is a short solution using two applications of Pascal's theorem.
26. H is the incenter of triangle DEF.

27. Find a nine-point circle.
28. What does the condition $\frac{AB}{AC} = \frac{BF}{FC}$ mean?
29. You can get away with applying Lemma 6.24 because $a\overline{a} = 1$.
30. Put ABC at the unit circle and compute points D, E directly.
31. Similarity generates some ratios.
32. How could we use the quantities $1 + ri$?
33. Construct the circumcenter of ABC and the midpoint of $\overline{AC}$. Do you see the three circles now?
34. Invert around A.
35. Combine Pascal's and La Hire's theorems.
36. Let T be the point on $\overline{AB}$ such that $AD = AT$.
37. Now we can just angle chase. Find the new cyclic quadrilateral.
38. Use some similar triangles to reduce this to Heron's formula.
39. Draw a very good diagram. You can construct A_2 as the second intersection of $\overline{TA_1}$ with Γ.
40. It equivalent to prove $\triangle CZM \sim \triangle EZP$. Hence all we want is $\angle CZE = \angle PZM$.
41. Construct the circle with diameter $\overline{AB}$.
42. Use a spiral similarity and do some computations.
43. Let $x = \angle ABQ$ and use trigonometry. Here $0^\circ < x < 60^\circ$.
44. $BE = BR = BC$.
45. What is the fixed point?
46. Show that $PD : AD = [PBC] : [ABC]$. Why are we done?
47. How can you map O to H using a homothety centered at G?
48. Ratios of the radii are sufficient.
49. What is the concurrency equivalent to?
50. Find some synthetic observations first. Parallel lines.
51. How do we handle the angle condition?
52. Deduce that the center of cyclic quadrilateral $BFEC$ must lie on $\overline{BC}$.
53. Add in the altitudes of ABC and compute a ratio.
54. Lemma 4.40 is likely to be very helpful.
55. Midpoints and parallel lines!
56. Intersect line EF with $\overline{AB}$ and $\overline{CD}$ to get tons of harmonic bundles.
57. This is pure angle chasing.
58. There is a very convenient point not marked that leads to a solution. Draw a good diagram.
59. Introduce the Miquel point M of the three circles.
60. Use the fact that $\angle B'OC' > \angle BOC$ to get $\angle A \le 60^\circ$.
61. Focus on triangle ACD.
62. Can you get rid of the points F and H in the expressions?
63. This follows since R is the intersection of $\overline{AC}$ and $\overline{BD}$.
64. Simson lines! Although angle chasing works as well.
65. Use both (e) and (f).
66. Note that B_1 is the intersection of lines C_1A_0 and AC, and the cyclic quadrilateral.
67. If $x = BD$, $y = AC$, and z is a third diagonal, one should obtain $xy = ac + bd$, $yz = ad + bc$, and $zx = ab + cd$.

68. The quantity $\frac{PA}{XY}$ does not depend on P.
69. Prove that $\measuredangle TLK = \measuredangle TCM$.
70. It is the circle with diameter $\overline{AB}$.
71. If all goes well, you should get something to the effect of $1 + \frac{1/2}{\sin(150^\circ - 2x)} = \frac{\sin x + \sin 60^\circ}{\sin(120^\circ - x)}$.
72. The condition is equivalent to the quadrilateral formed by lines KL, PQ, AB, AC being cyclic.
73. The two circles with diameter $\overline{AB}$ and $\overline{AC}$ hit the foot from A to $\overline{BC}$.
74. Look at triangle EBD. Notice anything familiar?
75. If $AB < AC$, show that M is an incenter.
76. Coaxial circles—show they have a second common point instead.
77. Show that $\measuredangle CMN = \measuredangle BMN$ first. (Another solution, perhaps more natural, begins by letting N' be the intersection of $\overline{AD}$ and $\overline{BC}$, and showing that N' lies on each of the circles.)
78. The strange part of the problem is the final condition $OP = OQ$, as the circumcenter is not related to anything in the problem. How might you encode this using something from this chapter?
79. Try point H.
80. Use Lemma 9.11 or Lemma 9.12.
81. Spiral similarity.
82. First show that B_1, B_2, C_1, C_2 are concyclic. What is their circumcenter?
83. It is negative since $\overline{AB}$ and $\overline{XY}$ are not disjoint.
84. Which quadrilateral is cyclic?
85. How can we get the orthocenter of AHE? We can do better than intersecting perpendicular lines.
86. Just expand $\frac{p-a}{p-b} \in \mathbb{R}$ directly.
87. It suffices to prove $OL \geq \frac{1}{2}R$. Can you think of some nice estimates for OL?
88. What do we know about the distance from O to all the sides?
89. The incenter/excenter lemma, see Lemma 1.18.
90. Show that the line is parallel to the Steiner line.
91. Exactly three of them have H as a vertex.
92. Spiral similarity is helpful here.
93. It is enough to show that the distance from O to $\overline{BC}$ is at least $\frac{1}{2}R$.
94. It suffices to show the circles are coaxial; equivalently, that they share the same radical axis. Use Lemma 7.24.
95. Find a Miquel point; then just angle chase.
96. We want to prove that $\angle OFB = \angle OGB = 90^\circ$. Invert around ω_1.
97. Add and subtract lengths to obtain $LH = XP$.
98. K is the Miquel point of a cyclic quadrilateral.
99. How else can we interpret the ray MH?
100. Show that $BQOP$ is cyclic.
101. Apply Lemma 1.18.
102. Use Menelaus's theorem.
103. Because A, I, I_A are collinear, just check that $\overline{AI_A} \perp \overline{I_BI_C}$.
104. The problem can now be solved with just two projections of harmonic bundles.

105. We obtain that H is the intersection of line AC and the line through B and the circumcenter of triangle ABC. Finish with the law of cosines.
106. Add a Miquel point.
107. Remember that you can project bundles on circles onto lines via points also on the circle.
108. Complete quadrilaterals.
109. $\measuredangle ABC = \measuredangle AA_1C$.
110. Shift O to O' and obtain a cyclic quadrilateral.
111. What does the tangency condition mean?
112. Invert again around ω_2!
113. First, show that $\overline{A_1B_1} \parallel \overline{A_2B_2}$. Then show that $\overline{A_1A_2}$, $\overline{B_1B_1}$, $\overline{C_1C_2}$ concur on Γ.
114. Rewrite the proof that a quadrilateral has angle sum 360° using directed angles.
115. Show that $ABOE$ is cyclic.
116. Power of a point.
117. Normally angle conditions are horrible. Why is this one okay?
118. Invert around D. The radius r can be anything.
119. Reflecting the orthocenter again.
120. I am sure you can guess which point to invert around.
121. You can shift M, N, H by $a + b + c$ before applying the circumcenter formula.
122. We have equal tangents at A.
123. First take the homotheties sending the squares outside the triangle.
124. You need two configurations. Use a good diagram to figure out what $\frac{HQ}{HR}$ should be.
125. $AXFEI$ is cyclic.
126. Let $D_1 = (u : m : n)$ and $A = (v : m : n)$, where D_1 is the second intersection of ω_1 and ω_2. This encodes all conditions.
127. Push the factor of 2 somewhere else.
128. The three concur at the symmedian point.
129. Now $\overline{AE}$ and $\overline{DB}$ are symmedians, so one can compute B, E. In addition, one can compute R as the intersection of the tangent at C and (the extension of) side AD.
130. $A^*B^* + B^*C^* \ge A^*C^*$ with equality if and only if A^*, B^*, C^* are collinear in that order. Now apply the inversion distance formula.
131. What must be true about the radical center?
132. Use the unit circle to get the orthocenter. $\frac{1}{2}(a + b + c + d)$.
133. First consider $X = P$ and $X = Q$; this gives four possible pairs (S, T).
134. Radical axes again.
135. Introduce the midpoint of EF to create a harmonic bundle involving S.
136. What is the equality case we are looking for?
137. The fixed point is the orthocenter.
138. Use a homothety.
139. It is also possible to compute the heights of the triangles.
140. This follows from the homothety used in the proof of Lemma 4.33.
141. Just compute all the points directly using (ABC) as the unit circle.
142. Try to show the contact triangle of ABC is homothetic to $\triangle PQL$.
143. Lemma 8.16 to clean up.

144. Use trigonometry to express the lengths BD and CE, which give the coordinates of D and E.
145. Midpoints and parallel lines.
146. Put $AB = 2x$, $CD = 2y$, $BC = 2\ell$ and compute some lengths.
147. Use Lemma 9.17 to compute the power of the midpoint. Then recall that all the centers are collinear.
148. Radical axes.
149. One should get $x = p + a + b + c - bc\overline{p}$.
150. Projective geometry.
151. Check that $\measuredangle YXP = \measuredangle AKP$.
152. You can replace line OH with any line through the centroid G.
153. Can you find a nice interpretation for the two given conditions?
154. Use a circle of radius zero.
155. Construct a rectangle. Show that the line through K^* perpendicular to $\overline{AQ}$ passes through the center of Γ.
156. There is something unnecessary in this problem.
157. Show that X, H, P are collinear, where P is said Miquel point.
158. Try homothety now.
159. Which quadrilateral is cyclic?
160. Recall Lemma 4.9.
161. The areas should come out to be $\frac{1}{8}ab\tan\frac{1}{2}C$.
162. Show that $[AOE] = [BOD]$ directly.
163. A is the Miquel point of BO_BO_CC.
164. Let $X = \overline{BE} \cap \overline{DF}$; by Lemma 9.18 we need $(X, H; E, F) = -1$.
165. What is the ratio of the homothety?
166. The given condition can be rewritten as $a^2 + c^2 - ac = b^2 + bd + d^2$.
167. We get $a_2 = \frac{bb_1 - cc_1}{b + b_1 - c - c_1}$, and then compute the determinant in Theorem 6.16.
168. Which quadrilateral is cyclic?
169. Show that the points lie on the circle with diameter $\overline{OP}$.
170. Inversion through D with radius 1.
171. Isogonal conjugates.
172. Invert around A.
173. Why does it suffice to consider the case $d = \overline{a}$, $e = \overline{b}$, $f = \overline{c}$?
174. Prove also that $\triangle EAB \cong \triangle MAB$.
175. Consider triangles XED and XAK.
176. Ignore $\triangle ABC$, and focus on $\triangle M_AM_BM_C$ instead. See if you can eliminate A, B, C from the picture entirely.
177. Try using power of a point.
178. The resulting four points should invert to something nice.
179. Find the fixed point first! A nice diagram helps here.
180. One can compute the numerical DF. Letting M be the midpoint of $\overline{DF}$, it suffices to show that $ME = \frac{1}{2}DF$.
181. We will be using AA similarity. Which angles are equal?
182. Begin by using cevian nest (Theorem 3.23).

183. Take a transformation that fixes $(ABCD)$ and sends Q to the center of the circle.
184. Use the law of sines.
185. First compute $\angle WXY = 40^\circ$.
186. Let O be the center of ω.
187. Use homothety.
188. Apply Theorem 7.14 directly to $AD = BE$ with reference triangle ABC.
189. Lemma 9.27 applies.
190. Begin by letting N be the point on $\overline{AK}$ so that $\overline{BN}$ is isogonal to $\overline{BC}$.
191. Rewrite the end condition without circles.
192. Show that line DT passes through the reflection of A over the perpendicular bisector of $\overline{BC}$.
193. The two circles intersect at the midpoint of major arc BC.
194. Use property (b) twice.
195. Show that $\angle AZY = \frac{1}{2}B$ and $\angle ZAX = \frac{1}{2}(A+C)$.
196. This problem is pretty silly.
197. A clean way to do this is by computing $[(a-b)(c-d)(e-f)+(b-c)(d-e)(f-a)]$ minus $[(a-b)(c-e)(d-f)+(d-e)(f-b)(a-c)]$.
198. An inversion can get rid of almost all the circles.
199. Where has this configuration appeared before?
200. If O_B and O_C are the centers, show that $O_BO_C = BC$.
201. Which quadrilateral is cyclic?
202. Law of cosines.
203. One should get $o_1 = \frac{c(a+c-2b)}{c-b}$ and $o_2 = \frac{b(a+b-2c)}{b-c}$. Now what is $\frac{1}{2}(o_1+o_2)$?
204. Show with computation that A, B_1, and C_1 are collinear. Then $\measuredangle C_1QP = \measuredangle ACP = \measuredangle AB_1P = \measuredangle C_1B_1P$.
205. Which configurations come to mind?
206. Pick reference triangle DEF. Here we pick $a = EF$, $b = FD$, $c = DE$.
207. A is the centroid of EBD, so ray DA bisects $\overline{BE}$.
208. Show that the ratio of the distance from Q to $\overline{AB}$ and $\overline{AC}$ is $AB : AC$. This will imply $\overline{AQ}$ is a symmedian.
209. Construct an isosceles trapezoid. Power of a point.
210. Use Lemma 6.18 in order to compute the points A_2, etc.
211. Prove that line PZ passes through the centers of ω and ω_1.
212. Find a good way to interpret the angle condition. Put another way, what are the possible locations of P?
213. Incenters.
214. $ADOO'$ and $BCOO'$ are also parallelograms.
215. Try erasing the points E, F, and A.
216. Show more strongly that if the intersection point is X, then $ABXC$ is harmonic.
217. The argument of $\frac{b-a}{c-a}$ is $\measuredangle BAC$, and the argument of $\frac{b-d}{c-d}$ is $\measuredangle BDC$.
218. Now $\overline{AB}$ and $\overline{CD}$ are diameters.
219. The two sides can be found to equal $\frac{BG \cdot CE}{BE \cdot CG}$.
220. Which quadrilateral is cyclic?
221. Why does it suffice to show $FBH'C$ is harmonic?

222. Consider the Gauss line of quadrilateral $ADBC$, and let M denote the midpoint of $\overline{EF}$.
223. This is one of my favorite tests of configuration recognition. You will need three of the lesser-used configurations.
224. One choice of reference triangle is $A_1A_2A_3$, with $A_4 = (p, q, r)$.
225. Use the law of sines.
226. You should get

$$\cos\left(\frac{3}{2}x + 30^\circ\right) = \cos\left(\frac{5}{2}x + 30^\circ\right) + \cos\left(\frac{1}{2}x + 30^\circ\right)$$

or some variant. One can guess the value of x now with some persistence (try multiples of 10°). Finish with sum-product on the right.
227. One standard trick for doing so: try to construct $\triangle A_2B_2C_2$ on Γ homothetic to $\triangle A_1B_1C_1$. Then show the center of homothety lies on Γ (implying it is T).
228. Using the fact that $\angle MEA = 90^\circ$, angle chase to show that $\overline{AF}$ is a symmedian.
229. Where does the isogonal conjugate of P lie?
230. You can explicitly find K.
231. Moreover, P is a point at infinity, so P, C, D collinear implies $ABCD$ is a square.
232. Let P be the midpoint of $\overline{QR}$ and L the midpoint of $\overline{MN}$. Show that $\overline{PK} \perp \overline{QR}$.
233. Brocard's theorem destroys this.
234. Take $ABCD$ to a rectangle; the problem becomes trivial.
235. Inversion at A with radius $s - a$ makes this much easier to compute. Overlays.
236. Just check that $\measuredangle MIT = -\measuredangle MKI$.
237. It is equal to $\frac{o_A - c}{b - c}$.
238. Consider the reflection of X, Y over $\overline{BC}$.
239. Now use Brocard's theorem.
240. Right angles and bisectors again.
241. This looks a lot like Pascal's theorem.
242. Show that $A_1^*A_2^*A_3^*$ and $B_1^*B_2^*B_3^*$ are homothetic (all sides parallel). Why is this enough?
243. Show that $N = (s - a : s - b : s - c)$. Normalize coordinates to check that $NG = 2GI$.
244. Homothety again.
245. Which quadrilateral is cyclic?
246. It is equivalent to show that $PC < PO$.
247. $A^*B^*C^*D^*$ is a parallelogram.
248. Add in the medial triangle.
249. You should be laughing.
250. Try to show the spiral similarity at K sends D to E as well; this implies the conclusion.
251. One should find that the resulting intersection is $(-a^2 : 2b^2 : 2c^2)$.
252. O is the reflection of C across $\overline{A^*B^*}$.
253. This is purely projective.
254. Let ray MH meet (ABC) again at K. It suffices to prove that $AKDE$ is cyclic.
255. What happens now if we invert about A?
256. Use area ratios on the inverted picture.

257. H and F swap places, as do A and E, as do C and F.
258. Now use Lemma 4.17.
259. Pick I the incenter of triangle BAD. Show that $IBCD$ is cyclic. Why does this solve the problem?
260. Which configuration is this?
261. Consider the second intersection of ω_1 and ω_2.
262. Try inverting.
263. Pin down Q by invoking Lemma 4.9.
264. The last ingredient is Lemma 4.17.
265. Now just angle chase.
266. Isogonal conjugates.
267. What is line QS?
268. It suffices to prove R, M, S are collinear.
269. Can you rephrase "$\overline{PH}$ bisects $\overline{EF}$" more naturally?
270. Brocard's theorem. Symmedians for the second part.
271. Try to compute N directly.
272. Do we want to deal with reflections? If not, what can we do?
273. It is equivalent to show that arcs $\widehat{TK}$ and $\widehat{TM}$ have the same measure.
274. Note that $\overline{CI} \perp \overline{A'B'}$ and $\overline{CM} \perp \overline{IK}$. What is the conclusion equivalent to?
275. Evaluate BE^2 in terms of a, b, c, using $\cos BAE = -\cos BAC$. Do the same for $\overline{AD}$ and then show $a^2 = b^2 + c^2$.
276. You can compute KN using $I_AN \cdot I_AK = I_AI^2 - r^2$.
277. Which quadrilateral is complete?
278. Show that $\measuredangle ZYP = \measuredangle XYP$.
279. Do not forget to preserve intersections of clines. For example, the circle tangent to ω should invert to a line tangent to ω at the same point.
280. One can compute $MS = MT$ explicitly. Just compute all the points directly.
281. You should obtain

$$J = \left(a\cos\left(A + \frac{1}{2}B\right) : b\cos\left(A + \frac{1}{2}B\right) : -c\cos\left(A - \frac{1}{2}B\right)\right)$$

or something similar.
282. First use homothety to make Q into something nicer.
283. Compute directly now; use A, S, T as free variables.
284. Use Ceva's theorem twice.
285. First show that $\overline{BC} \cap \overline{GE}$ lies on d.
286. The tangent at B is parallel to AP by angle chasing. Take perspectivity.
287. This just follows by taking the number line definition and solving $\frac{x-a}{x-b} : \frac{y-a}{y-b} = k$.
288. Draw a very good diagram. Can you say anything about the altitudes of $\triangle BHC$? (The next hint gives this away.)
289. We only care about the radical axis.
290. Let M be the midpoint of $\overline{BE}$. Show that $MA = ME = MB$.
291. One can also compute CR, say, by evaluating $AR = BR$ and applying Ptolemy's theorem.

292. It will reduce down to $(-a+b+c)(a-b+c)(a+b-c) \le abc$, called Schur's inequality.
293. Prove that A is the Miquel point of B_1BCC_1.
294. Try to get rid of a few circles.
295. We can find J by intersecting rays BJ and CJ.
296. Specifically, if $H_A = a+b+d$ is the orthocenter of $\triangle ABD$, then W is the midpoint of $\overline{AH_A}$.
297. Look at Lemma 1.44.
298. Show that the tangents to ω_1 and ω_2 meet on $\overline{BC}$.
299. You should get $\angle CXY = \angle AXP$ one way or another (good diagrams may suggest this as well). Use cyclic quadrilateral $APZX$ to prove this.
300. Find a harmonic bundle.
301. Look at all those circles. Can you get them to pass through more vertices?
302. Find a pair of similar triangles.
303. Now observe that X and Y are "$\pm\sqrt{de}$"; that is, $x+y=0$ and $xy=-de$. Moreover, show that $p^2 = de$.
304. The fixed point is $K = (2S_B, 2S_A : -c^2)$.
305. Which quadrilateral is cyclic?
306. You only need the fact that line TI passes through the midpoint of arc $\widehat{BC}$, say L.
307. Anything special on the median $\overline{EC}$?
308. First find the point of concurrency.
309. This yields Brocard's theorem.
310. Use Lemma 1.30 to handle the directed angles.
311. Find the diameter of the fixed circle.
312. The similarity is actually a congruence because $AC = BD$!
313. Are there some other reflections in this problem?
314. What is the orthocenter of the medial triangle?
315. If the problem is true, then the common radical axis must be the perpendicular bisector.
316. The key observation is that the circle is the midpoint of $\overline{AO}$.
317. Do you see an incenter?
318. The condition implies $DEBC$ is harmonic. What next?
319. Let $X = \overline{AD} \cap \overline{BC}$ and use Miquel points.
320. Do you recognize where the point D_2 has to be?
321. Use Lemma 8.16 applies directly.
322. The conditions should translate to $\angle D^*B^*C^* = 90^\circ$ and $B^*D^* = B^*C^*$.
323. Lemma 8.11.
324. In an overlaid picture, it suffices to show $\overline{MK^*}$ is tangent to the circumcircle of $\triangle K^*AQ$.
325. Draw a good diagram. Which three points look collinear?
326. What is $\angle AZY$?
327. $\triangle AO_BO_C \sim \triangle ABC$.
328. Letting E and F denote the tangency points of the incircle, we have $\overline{EF}$, $\overline{KL}$, $\overline{XY}$ concurrent now (due to the isosceles trapezoid).
329. Go via $\measuredangle AR^*B = \measuredangle AR^*O + \measuredangle OR^*B = \cdots = \measuredangle APB$.

330. Apply Brocard's theorem to locate H.
331. First identify B_1 and C_1.
332. Specifically, find the $\kappa \in \mathbb{R}$ such that $\kappa(a+b+c)$ lies on the Euler line of AIB (where $a = x^2$ and so on). Check that κ is symmetric in x, y, z.
333. Suppose the cevians meet at P. Where can we send A, B, C, P?
334. $m = 100$.
335. Then $\measuredangle FEM = \measuredangle FEB + \measuredangle BEM = \measuredangle FEB + \dots$?
336. This is essentially the same as the previous exercise.
337. Overlays are helpful here.
338. One should find that $\angle C^*B^*P^* = \angle B^*C^*P^*$. How to handle the incenters?
339. Areas.
340. Show that $\angle AA_1C_1$ is bisected by $\overline{A_1A_2}$. Thus P is the excenter of triangle A_1BC_1.
341. Why does $\angle AD^*B^* = \frac{1}{2}\angle AP^*B^*$?
342. Since $\angle M_CTA = \angle STM_B$, this is straight angle chasing now.
343. Why does it suffice to prove that $\frac{b}{c}\left(\frac{c-a}{b-a}\right)^2$ is real?
344. One can also get rid of A quickly. In other words, you can view the entire problem in terms of the quantities in quadrilateral $BGCE$.
345. Angle chasing can get rid of H and L completely.
346. By angle chasing, show that triangles MKL and APQ are similar. Why is this enough?
347. If E and F are the tangency points of the incircle and X is the second intersection of $\overline{AD}$ with the incircle, show that $DEXF$ is harmonic.
348. Just note that the side length of M_BM_C is half that of BC, so the ratio is -2.
349. For the setup, put $A = (au : bv : cw)$ and $C = (avw : bwu : cuv)$ and show that $PA = PC$ if and only if there is a common circle.
350. Homothety. Show that $O_BO_C = 2\left(\frac{1}{2}BC\right) = BC$.
351. Prove that $\overline{AD}$ is the polar of K.
352. Take a projective transformation, keeping Γ a circle. Many such transformations lead to a solution.
353. After the first inversion, we want to show that $\overline{F^*G^*}$ passes through B.
354. Extend ray IP to hit line BC at K. It suffices to show $(K, D; B, C) = -1$.
355. How do we use the condition that $AD = \frac{1}{2}AC$?
356. Let K' denote the intersection of the circumcircle and the angle bisector.
357. This is equivalent to $\frac{a-b}{p-q} : \frac{k-\ell}{a-c} \in \mathbb{R}$. Use Lemma 6.30 and expand.
358. Q is a Miquel point.
359. Borrow some ideas from the HMMT problem.
360. There is a homothety taking the medial triangle (the triangle whose vertices are the midpoints of ABC) to ABC itself. This follows from the opposite sides being parallel.
361. Identify the center of the circle first.
362. First get rid of Q by considering the point X diametrically opposite it on (ABC).
363. Note that $A_2A = PA$, where P is the tangency point of ℓ.
364. Show that the radical axis bisects $\angle PBC$.
365. Use $IE = x \sin C = \frac{cx}{2R}$ alongside Ptolemy's theorem to finish.
366. Radical axes give you a concurrence.

367. Consider the circles with diameters $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$.
368. Find Lemma 1.45 hidden in the picture.
369. Use isotomic conjugates and reflecting X, Y, Z, one can eliminate A, B, C altogether.
370. It is not hard to get $\tan \angle ZEP = \tan \angle ZCE = \frac{EZ}{ZC}$. So we just want to show $\frac{EZ}{CZ} = \frac{PE}{MC}$.
371. First compute d and e using Theorem 6.17. The hard part is computing o_1. You want a similar triangle.
372. Of course recall Lemma 1.18.
373. Show that both are equal to $90° - A$.
374. What is its center?
375. Find a Miquel point by using angle chasing.
376. Which quadrilateral is cyclic?
377. If $K = \overline{BB_1} \cap \overline{CC_1}$, prove B, K, A, C are concyclic.
378. Pascal's theorem on $AABBCC$.
379. There are symmedians in this problem.
380. Why might the quantity $\frac{1}{\sqrt{3}}(\cos 30° + \sin 30°)$ be useful?
381. The condition "ML tangent to (HMN)" is an abomination; perform some simplifying transformations.
382. M is the spiral center sending $\overline{YZ}$ to $\overline{BC}$.
383. Finish with the trigonemetric form of Ceva's theorem and the law of sines.
384. Invert around A.
385. Come on now, what configuration has that many perpendiculars?
386. Cut and paste!
387. Try to guess explicitly what A_2, B_2, C_2 are.
388. Because the triangles are easily similar (by angle chasing), focus on finding something shared by the two triangles.
389. Show that T^* and L^* are actually diametrically opposite on Γ^*.
390. This is just angle chasing.
391. You can compute $P_A Q_A$ in terms of ABC. Focus on just that.
392. I is the orthocenter of triangle BFC.
393. One should find $K = (a^2 : b^2 : c^2)$, $M = (0 : 1 : 1)$, and $L = (a^2 : S_C : S_B)$.
394. Dilate K and L and drop into a determinant.
395. Use Lemma 6.19 and do some calculations.
396. The use of "reflection" in this problem is kind of a misnomer. Draw a good diagram and you will see why.
397. Add a point to construct a cyclic quadrilateral.
398. Again just invert.
399. $\triangle BQM \sim \triangle NQC$, then use $BM : NC = AB : AC$.
400. Since $\overline{K^*M} \parallel \overline{AQ}$, it suffices to prove that $K^*A = K^*Q$.
401. This uses an idea similar to that of Problem 1.40.
402. Notice the duality between the nine-point circle and the circumcircle.
403. Inversion through the circle with diameter $\overline{AB}$ is most of the problem.
404. Construct a radical center.
405. Reflect the orthocenter.
406. Pascal's theorem on $AGEEBC$ first.

407. Law of sines.
408. What is the argument of $(1 + xi)(1 + yi)(1 + zi)$? Answer this in two ways.
409. H is a radical center.
410. Reflect the orthocenters.
411. $(A, B; X, Y) = -1 \Rightarrow (X, Y; A, B) = -1$.
412. More Simson line properties.
413. Reap the harmonic bundles using Lemma 9.17. You will want to use power of a point a lot.
414. Recall Theorem 2.25, the Pitot theorem.
415. Assume that $AB < AC$, and show that $\angle PQE = 90^\circ$.
416. Consider the radical axis of the circles with diameters $\overline{AB}$ and $\overline{CD}$.
417. Use the law of sines on $\triangle ABD$ and $\triangle ACD$.
418. Finish by taking a homothety to the centroid of $\triangle AST$, and finally to M.
419. The first part follows from Theorem 4.22.
420. Simson lines. Lemma 4.4 kills this.
421. After Pascal's theorem on $AABCCD$, we find that $\overline{AA} \cap \overline{CC}$ is collinear with $P = \overline{AB} \cap \overline{CD}$ and $Q = \overline{BC} \cap \overline{DA}$.
422. To handle the point T, use Lemma 4.40.
423. Add an incenter I.
424. Simson lines.
425. This is equivalent to showing A, E, S are collinear, where S and E are the reflection of T and D. Why does this follow from Lemma 4.40?
426. You want a homothety sending one of the points to another.
427. What to do with reflections?
428. Reuse the proof of Steiner lines.
429. Use the law of cosines to show the quadrilateral is cyclic, and then apply Theorem 5.10.
430. The fixed point is the orthocenter. Try reflecting the entire triangle.
431. Show that $\frac{p-(o_1+o_3)}{\overline{p}-(\overline{o}_1+\overline{o}_3)}$ is symmetric in a, b, c, d. It is easiest to evaluate the denominator first.
432. A, I, X are collinear. Hence we just want to show that $\overline{YZ} \perp \overline{AX}$ and the analogous equations.
433. Show that line NP passes through the circumcenter of triangle ABC.
434. Lemma 1.45.
435. How do we interpret the angle condition?
436. The condition $BC = DA$, $BE = DF$ can be weakened to just $\frac{BE}{BC} = \frac{DF}{DA}$.
437. Actually, you do not even need ID, IE. The answer is no.
438. Finish off with Lemma 4.14.
439. All circles pass through one point.
440. Show that P is the desired incenter.
441. You can simplify $\sin x + \sin 60^\circ$ to cancel with something in the denominator.
442. First get rid of the midpoints of the altitudes using Lemma 4.14. Who uses midpoints of altitudes?
443. Brocard's theorem on $ABCD$, $AGCH$, with K the radical center of the three circles.

444. The condition that $AEDC$ is cyclic is actually extraneous! What does this allow us to do?
445. One should compute the circumcenter as $\frac{(a+b+c)(b^2+c^2)}{b^2+bc+c^2}$.
446. Ceva's theorem with a quick angle chase.
447. One can compute the points K, G, T first, then use symmetry.
448. Use law of sines on the five triangles. Vertical angles cancel.
449. Take the tangency point of the A-excircle as Q_1. Ignore Q now.
450. Let $\overline{KI_A}$ (with I_A the A-excenter) meet the perpendicular bisector of $\overline{BC}$ at T. Show that $BNCT$ is cyclic.
451. Project it through E.
452. Repeatedly use law of sines and power of a point.
453. Which quadrilateral is cyclic?
454. Note that Lemma 1.17 helps involve $\overline{HM}$.
455. How can one obtain angle information from midpoints?
456. Try sending the points $\overline{AB} \cap \overline{XY}$, $\overline{BC} \cap \overline{YZ}$ infinitely far away.
457. First compute PK and QL.
458. In Figure 4.2A, consider the midpoint of $\overline{II_A}$.
459. Which quadrilateral is cyclic?
460. Let I be the incenter.
461. Use Theorem 7.25 now to handle the circumcenter.
462. Write this as $[ABC] = [AIB] + [BIC] + [CIA]$, with I the incenter.
463. Answer is $(c^2 : b^2 : c^2)$, up to scaling.
464. See if you can guess the fixed point. (Pick a convenient P.)
465. Use Lemma 8.10.
466. Now use Conway's formula (Theorem 7.22).
467. Sum equal tangents.
468. Power of a point.
469. Symmedians.
470. Note that $\overline{AI}$ bisects $\angle B'AC'$.
471. Prove $(A, D; M, N) = -1$.
472. This is pure angle chasing.
473. After both applications, we find that $\overline{AA} \cap \overline{CC}$, $\overline{BB} \cap \overline{DD}$, P, Q are collinear.
474. Let T be the intersection of the tangents at A and K. Show that $ATKM$ is cyclic and recall $TK = TA$.
475. Several forms of computation work, but there is a very clean solution.
476. First compute $\angle CYX$ in terms of angles at X. What you get depends on what variables you selected.
477. Ptolemy's theorem.
478. Use Ceva's theorem to show that ray AP bisects the opposite side.
479. Answers are 30° and 150°.
480. It just gives a pair of similar triangles.
481. The common point is the Miquel point M of $ADBC$.
482. The perpendicular bisectors are actually just giving you a circumcenter.
483. Some lengths in the figure are computable. Let $AC = 3$ and compute some lengths.
484. Lemma 1.45.

485. Try adding the circumcenter O.
486. Compute the lengths BP, CP, BQ, CQ using similar triangles, and then compute all points directly.
487. M is the center of the spiral similarity sending $\overline{AB}$ to $\overline{CD}$, so it also sends O_1 to O_2.
488. The determinant can be rewritten so that all terms are degree 2.
489. Just some angle chasing with the above.
490. Show without barycentrics that the cevians concur. Name the concurrency point.
491. Homothety with ratio $\frac{1}{2}$.
492. By Brocard's theorem, $\overline{EF} \cap \overline{BC}$ has polar $\overline{AH}$.
493. Spiral similarity sending $\overline{AD}$ to $\overline{BC}$ also sends E to F.
494. It suffices to prove that $\overline{MN} \parallel \overline{AD}$. (Why?)
495. There is a radical axis.
496. Just use Lemma 1.48 now.
497. Then Pascal's theorem on $CG'GEBB$, where G' is the reflection.
498. What technique does this lemma open up that was not feasible before?
499. What is the Miquel point of complete quadrilateral $FARM$?
500. This is true whenever $A \le 60^\circ$. Prove this.
501. Add a nine-point circle!
502. There are three cyclic quadrilaterals due to all the right angles, as well as $ABPC$ itself. Use these to your advantage.
503. Let T be the intersection of line EF with $\overline{CD}$. Show that T lies on (ABM).
504. Show that D, P, E are collinear, and angle chase.
505. I is the orthocenter of $\triangle BHC$. Use Lemma 4.6.
506. Suppose we wish to show $\angle BOC = 2\angle BAC$. Put A, B, C on the unit circle.
507. Use Lemma 1.45 to handle the nine-point circle.
508. This just follows from the homothety between ABC and $AB'C'$ sending E to X.
509. How can we compute A_2 nicely?
510. Use Lemma 1.44.
511. There are three circles through one point. What might this motivate you to do?
512. Let X, Y denote the midpoints of $\overline{BD}$ and $\overline{CE}$. Show that IM is the line through I perpendicular to the Gauss line XY.
513. At this point $s = b + c - abc$ and so on. Apply Theorem 6.15.
514. There is a homothety between triangles $I_A I_B I_C$ and DEF.
515. One should get $a^2 - ac + c^2 = \frac{(ab+cd)(ad+bc)}{ac+bd}$.
516. Where is H?
517. Look for spiral similarities with (ADM) and (ABC).
518. Use reference triangle PBC.
519. Apply Lemma 4.4 directly, using a homothety with ratio 2.
520. Note that $ABCD$ is harmonic, so $(A, C; B, D) = -1$; projecting through E gives that $(A, C; \overline{BE} \cap \overline{AC}, P_\infty) = -1$, where P_∞ is the point at infinity along line AC.
521. This is obvious by Lemma 1.17.
522. Use the law of cosines now and some trigonometry. PO can be found by the law of cosines on $\triangle PCO$.
523. Take $WXYZ$ with $WX = a$, $XY = c$, $YZ = b$, $ZW = d$. Find WY.

524. Use triangle ACD as the reference triangle.
525. Q is a Miquel point of quadrilateral $DXAP$.
526. Consider the four tangency points W, X, Y, Z and solve the problem in terms of them.
527. The radical center is N.
528. Isogonal conjugates.
529. Hidden symmetry.
530. Let A_1 be the point diametrically opposite A on the circle.
531. The first part is relatively easy angle chasing, the second part is fairly short complex numbers.
532. What is the line G_1 and I?
533. Focus on $\triangle AST$; points P and Q are not especially important.
534. Specifically: construct $\overline{AB} \cap \overline{CD}$ and $\overline{BC} \cap \overline{DA}$. Do you notice anything?
535. A solution to this exercise appears as a linear algebra example in Appendix A.1.
536. After a homothety on the inverted picture, does this look familiar?
537. If the four points are not concyclic, what point must the radical axis of (PRS) and (QRS) pass through?
538. K is the incenter of $\triangle LED$.
539. What do A^*, B^*, C^* look like at the equality case when $ABCD$ is cyclic?
540. Work with each center individually.
541. You can just angle chase this one.
542. Take a homothety.
543. First recall Lemma 4.17.
544. The condition $OP = OQ$ is equivalent $R^2 - OP^2 = R^2 - OQ^2$.
545. Use the fact that $AG = 2GM$.
546. Apply barycentric coordinates to the resulting problem.
547. What is the best way to characterize the Euler lines of the other triangles?
548. The point of concurrency is yet another radical center.
549. Avoid intersecting quadratics. Find a better way.
550. What is $OA_1 \cdot OA_2$ in terms of the circumradius R?
551. What is the orthocenter of $\triangle CIK$?
552. You can compute everything.
553. Show the circles are coaxial by finding a second point with the same power to all the circles. Why does this imply the conclusion?
554. Use $\triangle AOD \sim \triangle DCO_1$ to get $\frac{o_1-d}{c-d} = \frac{o-d}{a-d}$, and then compute o_1.
555. Construct a quadrilateral.
556. $\measuredangle HSR = \measuredangle HBC$ by spiral similarity, but $\measuredangle HBC = \measuredangle HSM$ as well.
557. The tangents from P to this circle lie on a line through X. Now just apply similar triangles and/or power of a point.
558. The center of $\triangle O_A O_B O_C$ is $\frac{o_A+o_B+o_C}{3}$. Note that we do not need the unit circle at all in this problem.
559. Trigonometry will work, but there is an elegant synthetic solution.
560. Simply verify that each of A^*, B^*, C^* lies on the nine-point circle.
561. A_1^* is the midpoint of $\overline{EF}$, etc. The three circles are congruent, so C_1^* is parallel to $\overline{EF}$.

562. Focus on the conditions $BC = DA$ and $BE = DF$. (These can actually be weakened.)
563. Start from $(A, Z; K, L) = -1$; end with M the midpoint of $\overline{PQ}$. Here Z is the concurrency point of $\overline{EF}$, $\overline{KL}$, $\overline{XY}$.
564. This is just angle chasing.
565. Express BC^2 in two ways.
566. Try inverting through the incircle.
567. There are still degrees of freedom left. How might we handle them?
568. Find a hidden circle.
569. Try using Example 1.4.
570. Show that $HM \cdot HP = HN \cdot HQ$.
571. Take perspectivity at C onto k.
572. Here is one finish: let $T = \overline{AD} \cap \overline{CE}$ and send $\overline{BT} \cap \overline{AC}$ to the center of Γ.
573. Complete the quadrilateral. (Trigonometry also works.)
574. Points M and N can be computed by normalizing coordinates and then using $\vec{M} = 2\vec{P} - \vec{A}$.
575. Add in the center O. Which quadrilateral is cyclic?
576. It suffices to show $\overline{MN} \parallel \overline{AD}$.
577. The inverted image should be a rectangle.
578. Inversion around (DEF) once more. Use Lemma 8.11 again.
579. We do not know where O^* goes, but we only care that the center of $(A^*B^*C^*)$ lies on the Euler line of the contact triangle, since this center is collinear with I and O. Why is this obvious?
580. Spiral similarities come in pairs.
581. Again, inversion to eliminate the strange angle condition.
582. Look for harmonic bundles involving T and lines XY and BC.
583. Reflect B over M in order.
584. Combine this with (d) to show that N is a midpoint.
585. Draw a good diagram. Something should appear readily.
586. Line through circumcenter and centroid of AIB.
587. Complete the quadrilateral.
588. Now use Lemma 7.23.
589. Just consider $(1 + x_1 i)(1 + x_2 i) \dots (1 + x_n i)$.
590. Apply Brocard's theorem repeatedly.
591. What is $\frac{\sin \angle BAD}{\sin \angle CAD}$?
592. You have a cyclic trapezoid; hence it is isosceles.
593. Which quadrilateral is cyclic?
594. The symmedian is isogonal to the midpoint.
595. Make ABC an equilateral triangle and with center P. Use Lemma 9.8.
596. How do we handle the bisector condition?
597. Which radical axis passes through A?
598. Without loss of generality, B, C lie on the same side of the line. Let M be the midpoint of $\overline{BC}$.
599. This is just a statement about distances to line OH; ignore the areas.

600. How do we handle the reflection?
601. Observe that $\overline{AB}$ is a tangent to (PRS).
602. The desired concurrency point is the isogonal conjugate of the Nagel point. The calculations can be made very clean.
603. Use the law of cosines.
604. Use the spiral similarity at X to handle the midpoints. Push N to M. Then angle chase to compute $\measuredangle NMX$.
605. The area of triangle BIC is $\frac{1}{2}ar$.
606. Use the ratio $\frac{BC_1}{CB_1}$ as a proxy.
607. Note $ABCD$ is a harmonic quadrilateral.
608. Compute $|p-x||p-y|$ directly. The answer is BC^2.
609. You want $\overline{PH}$ to pass through the foot from I to $\overline{EF}$. Several of the points are extraneous now.
610. Letting $x = ID = BD = CD$, what is IE?
611. Again radical centers.
612. Isosceles triangles should appear.
613. We want to use the trigonometric form of Ceva's theorem to show the conclusion, since the intersection $\overline{AD} \cap \overline{BC}$ seems fairly random.
614. Show that $ABCD$ is cyclic.
615. Q is a Miquel point.
616. If O is the center of ω, let $\overline{OP}$ meet ω again at X. Power of a point now.
617. First get rid of S and T.
618. Those squares inside the triangle are weird. Can we make them nicer?
619. What happens in the limiting case $\angle A + \angle COP = 90^\circ$? Do you notice anything?
620. The inverses of the sides of $A_1A_2A_3$ are the circles with diameter $\overline{ID}$, $\overline{IE}$, $\overline{IF}$, where D, E, F are the tangency points.
621. Put $T = a^2qr + b^2rp + c^2pq$ to simplify calculations.
622. This is asking for trigonometry. The extended law of sines is helpful because everything is in a central circle, and right angles are everywhere. There are two degrees of freedom.
623. $(\frac{1}{2}, \frac{1}{2}, 0)$, or equivalently $(1:1:0)$. The latter is usually easier to work with for computations.
624. Just apply a couple homotheties now.
625. Consider the circle with diameter $\overline{BC}$.
626. Try inverting around C.
627. Show that the quadrilateral formed by lines EF, GH, AB, CD is cyclic (power of a point at $\overline{AB} \cap \overline{CD}$).
628. Prove a more general version of (b).
629. There are three circles with a useful radical center.
630. Prove that the center of the spiral similarity taking $\overline{BD}$ to $\overline{CE}$ is M.
631. Trignometric form of Ceva's theorem.
632. Complete the Brocard configuration. Note $\overline{OM} \perp \overline{CD}$.
633. Spiral similarity at H.
634. Begin with Lemma 4.14 and Lemma 4.33.

635. What is the conclusion equivalent to?
636. Note if you haven't already that $\overline{AP}$ is a median, so we wish to show $\overline{AQ}$ is a symmedian.
637. Can you find a way to use the isosceles triangles?
638. Invert around A.
639. Show that the nine-point center moves on a circle centered at A.
640. What is K?
641. Just use reference triangle PBD to handle the conjugates.
642. M is the reflection of B across $\overline{CH}$.
643. Where has this point O come up before?
644. This is just column operations in the determinant.
645. $\measuredangle DAB = \measuredangle DAC + \measuredangle CAB$ and $\measuredangle BCD = \measuredangle BCA + \measuredangle ACD$.
646. Begin by using part (d) of Lemma 4.40.
647. The condition $\angle BAG = \angle CAX$ just means the fixed point has the form $(k : b^2 : c^2)$ (symmedians). Use this to your advantage.
648. What happens under inversion at A?
649. It should be 1. Now show that $(a-b)(c-e)(d-f) + (d-e)(f-b)(a-c) = 0$.
650. A complex number $1 + i\tan(\theta)$ has argument θ.
651. Pick a reference triangle that makes the circles nice.
652. All the points have decent closed forms. Just compute the determinant.
653. You are asked to show the fixed point has form $(m : 1 : 1)$. Use this to your advantage by computing m and showing it does not depend on u or v
654. What point has equal power to both circles?
655. Add in the circumcenter O.
656. The rest is computation. One working setup is $\alpha = \angle CXY = \angle AXB$, $\beta = \angle BXY$.
657. Let $D = (0 : u : v)$ with $u + v = a$ and compute the circles directly.
658. Find some more bisected angles.
659. Which quadrilateral is cyclic?
660. Show that $PEDQ$, $QFER$, $PFDR$ are all cyclic.
661. Simson lines from Y might help (but the problem can be solved without them). For the other solution, begin by noting the desired angle is $\angle PQY + \angle SRY - \angle QYR$.
662. Translate the condition $MB \cdot MD = MC^2$.
663. Nine-point circles.
664. Look for an angle bisector, and prove it using barycentrics. Finish from there.
665. Take a homothety which sends the square outside.
666. It is simply $\frac{XB}{XA}$ (directed). This follows from $\frac{P_\infty B}{P_\infty A} = 1$.
667. This problem is purely projective.
668. Compute $\frac{b-a}{f-a} \cdot \frac{d-c}{b-c} \cdot \frac{f-e}{d-e}$.
669. Length chasing and similar triangles work.
670. After finding the cyclic quadrilateral, apply Lemma 1.18.
671. The centroid G is the weird guy. How do we handle it?
672. Recall Lemma 4.33. How is $\overline{ZM}$ related to the circles?
673. Do a negative inversion through H mapping the nine-point circle to the circumcircle.
674. Notice first that $HBYC$ is a parallelogram (because of the midpoints).
675. After adding in the point diametrically opposite B, use Pascal's theorem.

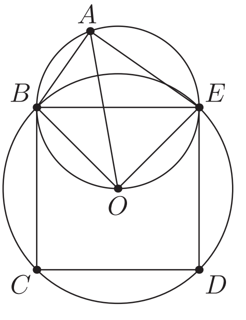
676. Complement Lemma 4.33 by extending AO to meet Γ again.
677. Try to get parallel lines instead of tangency.
678. Just use the $\frac{1}{2}ab\sin C$ formula.
679. Inversion around B seems nicest (many lines through B).
680. Get another pair of similar triangles and then angle chase to finish.
681. Simson lines.
682. A certain configuration is quite helpful here.
683. Ceva's theorem combined with Lemma 2.15.
684. You will need to halve angles. Do not use directed angles; the problem is false if A, C, B, D lie in that order.
685. Let $A_1B_1C_1$ be the determined triangle, and let T be the tangency point. How might you show tangency of two circles?
686. It suffices to show that this spiral similarity also sends X to P. Just show $\measuredangle MXY = \measuredangle MPB$.
687. Midpoints and parallel lines!
688. Plug in $A = (1, 0, 0)$, to get $u = 0$, then do the same with B and C.
689. Let $\overline{AD}$ meet the incircle again at X. Can you find a harmonic quadrilateral?
690. Try to show that E lies on a circle with diameter $\overline{DF}$.
691. Draw a good diagram. What is the relation of A_2, B, C to (ABC)?
692. Steiner line of complete quadrilateral $BEDC$.
693. Let O be the circumcenter of ABD. Show that $ODCF$ is a parallelogram. Then note $OA = OB = OD = 1$.
694. Show that when inverting with radius $\sqrt{BH \cdot BE}$, P and Q are inverses.

APPENDIX C

Selected Solutions

C.1 Solutions to Chapters 1–4

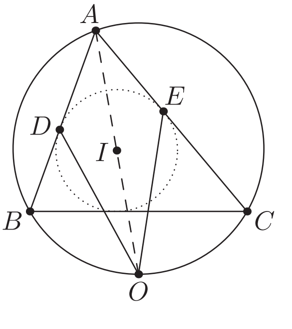
Solution 1.36



Observe that $\angle BAE = 90^\circ$ and $\angle BOE = 90^\circ$. It follows that $ABOE$ is cyclic. So $\angle OAE = \angle OBE = 45^\circ$ and $\angle BAO = \angle BEO = 45^\circ$. It follows that $\angle OAE = \angle BAO = 45^\circ$, as needed.

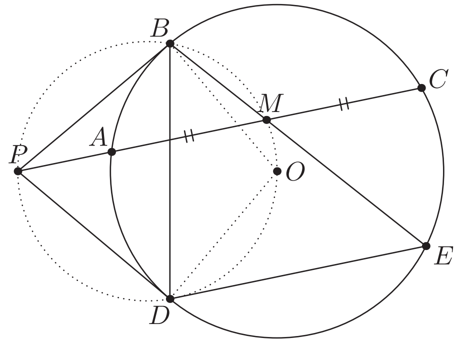
The condition that $ABCDE$ is convex ensures that A lies on the opposite side of $\overline{BE}$ as O, so there is no need to worry about configuration issues and it is fine to just use standard angles.

Solution 1.39



By Lemma 1.18, O lies on line AI. Now AI is an angle bisector and $AD = AE$, so it follows that $\triangle ADO \cong \triangle AEO$, so $\angle ADO = \angle AEO$ and hence $\angle BDO = \angle OEC$.

Solution 1.43



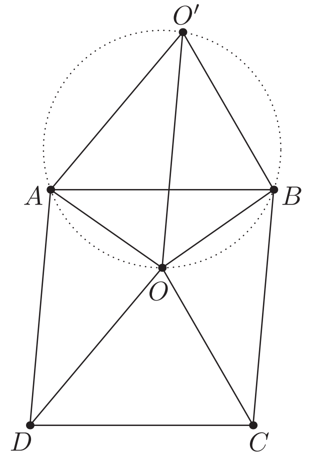
Let M be the intersection point of $\overline{BE}$ and $\overline{AC}$. We wish to show that $\overline{OM} \perp \overline{AC}$. Since $\measuredangle PBO = \measuredangle PDO = 90^\circ$, points P, B, D, O are concyclic.

We claim that M lies on this circle too. Indeed, since $\overline{DE} \parallel \overline{AC}$ we have

$$\measuredangle BMP = \measuredangle BMA = \measuredangle BED = \measuredangle PBD = \measuredangle BDP.$$

Consequently, $\measuredangle OMP = \measuredangle OBP = 90^\circ$ as desired.

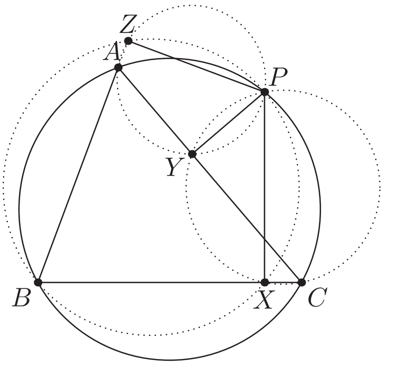
Solution 1.46



Let O' be a point such that $DAO'O$ is a parallelogram. Since $OO' = DA = BC$ and all three lines are parallel, it follows that $CBO'O$ is a parallelogram as well. Moreover, we have $\angle AO'B = \angle DOC$, since $\overline{AO'} \parallel DO$ and $\overline{BO'} \parallel \overline{CO}$. Consequently, $\angle AO'B + \angle AOB = 180^\circ$ and $AO'BO$ is cyclic (note that O' must lie outside the parallelogram since O is given to lie inside it). Actually, one can even check that $\triangle O'AB \cong \triangle OBC$.

Consequently, $\angle CBO = \angle O'OB = \angle O'AB = \angle ODC$ as needed.

Solution 1.48

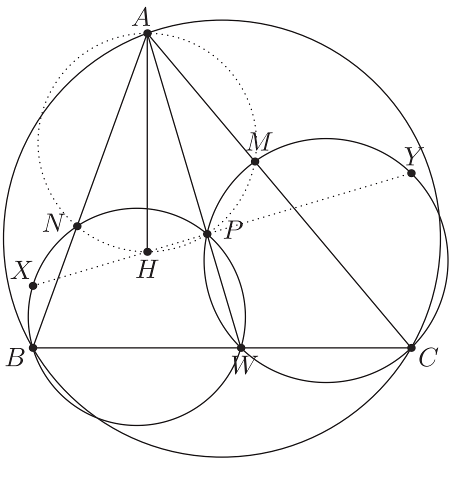


The main observation is that all the altitudes produce cyclic quadrilaterals: P lies on the circumcircle of all three triangles YZA, ZXB, and XYC. Hence we can directly compute

$$\measuredangle PYZ = \measuredangle PAZ = \measuredangle PAB = \measuredangle PCB = \measuredangle PCX = \measuredangle PYX.$$

This implies X, Y, Z are collinear.

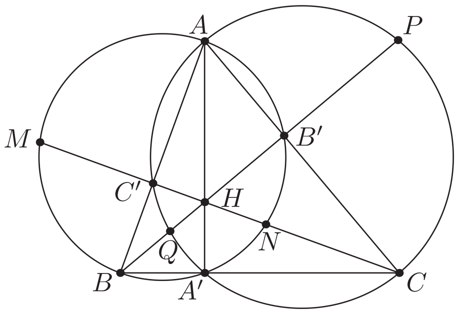
Solution 1.50



Let P be the second intersection of ω_1 and ω_2. By Lemma 1.27, we have that P also lies on the circumcircle of triangle AMN. But recall by Lemma 1.14 that this is the circle with diameter $\overline{AH}$. It follows that $\angle APH = 90^\circ$.

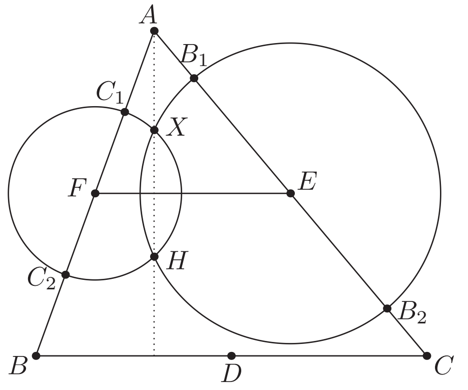
Now, observe that $\angle XPW = 90^\circ$ by construction. We find that X, H, P are collinear. Similarly, Y, H, P are collinear. Therefore, X, Y, H are collinear.

Solution 2.26



Let A' be the foot of the altitude from A to $\overline{BC}$, and notice that A' lies on both the circles in the problem. Now we can apply Theorem 2.9 directly. The radical center is the orthocenter H of the triangle.

Solution 2.29



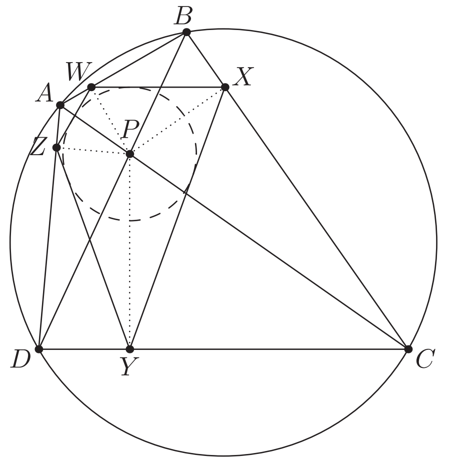
Let D, E, F be the centers of Γ_A, Γ_B, Γ_C.

We first show that B_1, B_2, C_1, C_2 are concyclic. By Theorem 2.9, it suffices to prove that A lies on the radical axis of the circles Γ_B and Γ_C.

Let X be the second intersection of Γ_B and Γ_C. Clearly $\overline{XH}$ is perpendicular to the line joining the centers of the circles, namely $\overline{EF}$. But $\overline{EF} \parallel \overline{BC}$, so $\overline{XH} \perp \overline{BC}$. Since $\overline{AH} \perp \overline{BC}$ as well, we find that A, X, H are collinear, as needed.

Thus, B_1, B_2, C_1, C_2 are concyclic. Now their circumcenter is the intersection of the perpendicular bisectors of $\overline{C_1C_2}$ and $\overline{B_1B_2}$, which is none other than the circumcenter O of ABC. Hence what we have proved is that $OB_1 = OB_2 = OC_1 = OC_2$. Similarly we can prove $OA_1 = OA_2 = OB_1 = OB_2$ and the proof is complete.

Solution 2.34

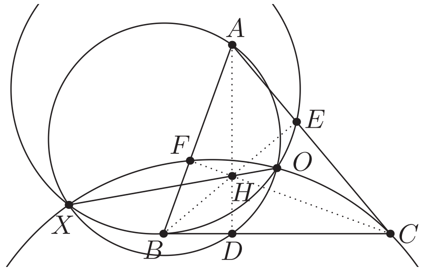


Since $\measuredangle AWP = \measuredangle AZP = 90^\circ$, we have that $AWPZ$ is cyclic. Similarly, so is $BWPX$. Hence,

$$\measuredangle ZWP = \measuredangle ZAP = \measuredangle DAC = \measuredangle DBC = \measuredangle PBX = \measuredangle PWX.$$

Therefore, P lies on the angle bisector of $\angle XWZ$. Similarly, it also lies on the angle bisectors of $\angle WZY$, $\angle ZYX$, and $\angle YXW$. Hence the distance from P to each side of $WXYZ$ is the same, and we can draw a circle centered at P tangent to all four sides. The conclusion of the problem then follows from Theorem 2.25.

Solution 2.36

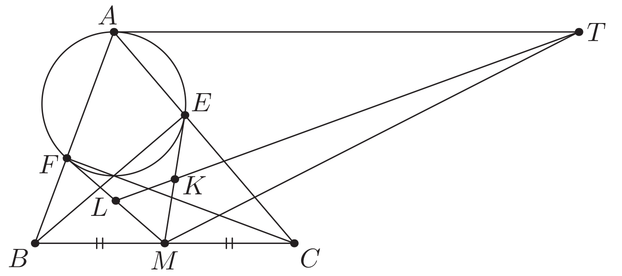


Let H be the orthocenter of ABC. Let $\omega_A, \omega_B, \omega_C$ denote the circumcircles of triangles AOD, BOE, COF, respectively. Let X be the second intersection of ω_A and ω_B. Evidently the radical axis of ω_A and ω_B is line XO.

By considering the circles with diameters $\overline{BC}$, $\overline{CA}$, $\overline{AB}$, we find $AH \cdot HD = BH \cdot HE = CH \cdot HF$. So H has equal power with respect to all three circles. Since H and O are distinct, that means H lies on line XO, It also implies that line HO is the radical axis of ω_B and ω_C.

Since X, O, H are collinear, we find X lies on the radical axis of ω_B and ω_C. But X has power zero with respect to ω_B. Hence it also has power zero with respect to ω_C. So X lies on ω_C as well.

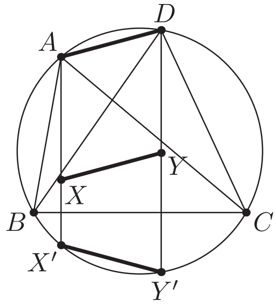
Solution 2.38



Let ω denote the circumcircle of $\triangle AEF$. Recall by Lemma 1.44 that $\overline{TA}$, $\overline{MF}$, $\overline{ME}$ are all tangents to the circumcircle of ω. Now consider the circle ω as well as the circle γ_0 centered at M with radius zero. Notice that K lies on the radical axis of ω and γ_0, since $\text{Pow}_\omega(K) = KE^2 = KM^2 = \text{Pow}_{\gamma_0}(K)$. Similarly, L lies on the radical axis as well. Hence, KL is the radical axis of these two circles.

Then $TA^2 = \text{Pow}_\omega(T) = \text{Pow}_{\gamma_0}(T) = TM^2$, so $TA = TM$.

Solution 3.17

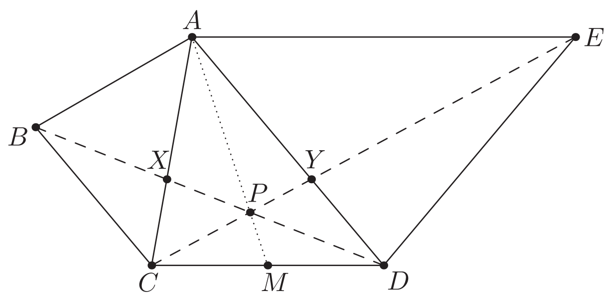


Let the reflections of X and Y over $\overline{BC}$ be X' and Y'. As we have reflected the orthocenters over the sides, by Lemma 1.17 we find that X' and Y' lie on the circumcircle ω of $ABCD$.

Thus we find that $X'Y' = XY$. It is also clear that $\overline{AX'} \parallel \overline{DY'}$. Therefore, we have a cyclic trapezoid $AX'Y'D$, meaning $X'Y' = AD$ as well. Consequently, $AD = XY$.

Therefore, we have $\overline{AX} \parallel \overline{DY}$ and $AD = XY$. Hence $AXYD$ is either a parallelogram or a trapezoid. Actually, since $\overline{AD}$ is the reflection of $\overline{X_1Y_1}$ across the diameter of ω parallel to $\overline{BC}$, while $\overline{XY}$ is the reflection of $\overline{X_1Y_1}$ over $\overline{BC}$, it follows that we must be in the parallelogram case.

Solution 3.19

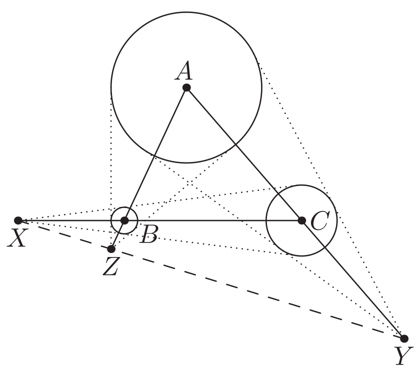


Let X denote the intersection of diagonals $\overline{AC}$ and $\overline{BD}$. Let Y denote the intersection of diagonals $\overline{AD}$ and $\overline{CE}$.

The given conditions imply that $\triangle ABC \sim \triangle ACD \sim \triangle ADE$. From this it follows that quadrilaterals $ABCD$ and $ACDE$ are similar. In particular, we have that $\frac{AX}{XC} = \frac{AY}{YD}$.

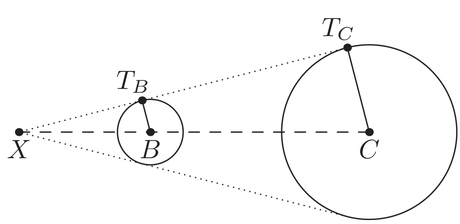
Now let ray AP meet $\overline{CD}$ at M. Then Ceva's theorem applied to triangle ACD implies that $\frac{AX}{XC} \cdot \frac{CM}{MD} \cdot \frac{DY}{YA} = 1$, so $CM = MD$.

Solution 3.22



Let the centers of the circles be A, B, C and denote the radii by r_a, r_b, r_c. Let the tangents for the circles centered at B and C meet at X. Define Y and Z analogously.

It is not hard to check that X lies outside $\overline{BC}$. Consider the similar right triangles exhibited below.



We see that

$$\left|\frac{XB}{XC}\right| = \frac{r_b}{r_c}.$$

Hence, in the notation of Menelaus's theorem, we have

$$\frac{BX}{XC} = -\frac{r_b}{r_c}.$$

Analogously, we have $\frac{CY}{YA} = -\frac{r_c}{r_a}$ and $\frac{AZ}{ZB} = -\frac{r_a}{r_b}$. So

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = -1$$

as needed.

Solution 3.23

Refer to Figure 3.7B. By the law of sines, we have

$$\frac{\sin \angle BAD}{\sin \angle CAD} = \frac{\frac{ZD}{ZA} \sin \angle ADZ}{\frac{YD}{YA} \sin \angle ADY} = \frac{ZD}{YD} \cdot \frac{YA}{ZA}.$$

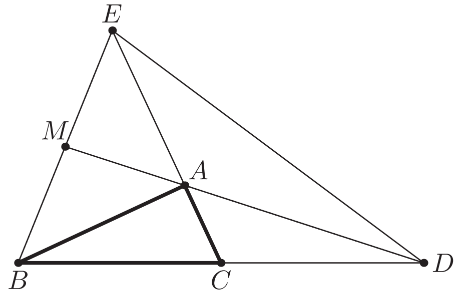
So by Ceva's theorem in trigonometric form, it suffices to prove that

$$\left(\frac{ZD}{YD} \cdot \frac{YA}{ZA}\right)\left(\frac{XE}{ZE} \cdot \frac{ZB}{XB}\right)\left(\frac{YF}{XF} \cdot \frac{XC}{YC}\right) = 1.$$

But this follows by noting that Ceva's theorem on $\triangle XYZ$ and $\triangle ABC$ gives us

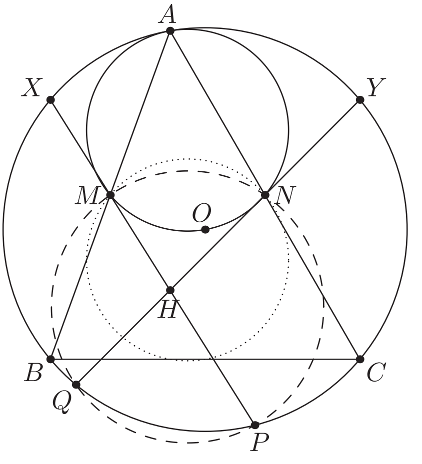
$$\frac{ZD}{YD} \cdot \frac{YF}{XF} \cdot \frac{XE}{ZE} = \frac{ZB}{ZA} \cdot \frac{YA}{YC} \cdot \frac{XC}{XB} = 1.$$

Solution 3.26



Let ray DA meet $\overline{BE}$ at M. Consider the triangle EBD. Since the point lies on median $\overline{EC}$, and $EA = 2AC$, it follows that A is the centroid of $\triangle EBD$. So M is the midpoint of $\overline{BE}$. Moreover $MA = \frac{1}{2}AD = \frac{1}{2}BE$; so $MA = MB = ME$ and hence $\triangle ABE$ is inscribed in a circle with diameter $\overline{BE}$. Thus $\angle BAE = 90^\circ$, so $\angle BAC = 90^\circ$.

Solution 3.29



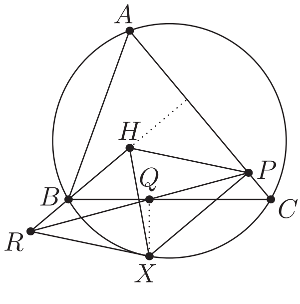
The main point of the problem is actually to prove that M, N, P, Q are concyclic. Then we can apply radical axis to the circles (AMN), (ABC), and $(MNPQ)$ to deduce that their radical center is the point R described in the problem (not shown in the figure).

Suppose the homothety taking the nine-point circle of ABC to the circumcircle of ABC itself sends M and N to points X and Y on the circumcircle of ABC. Put another way, let X and Y denote the reflections of H over M and N. By power of a point, we know that $XH \cdot HP = YH \cdot HQ$. Since $MH = \frac{1}{2}XH$ and $NH = \frac{1}{2}YH$, it follows that $MH \cdot HP = NH \cdot HQ$, and the problem is solved.

Solution 4.42

Let ω be the circumcircle of ABC. By Lemma 1.18, the circumcenter of $\triangle IAB$ lies on ω. So do the circumcenters of $\triangle IBC$ and $\triangle ICA$. Hence ω is the requested circle.

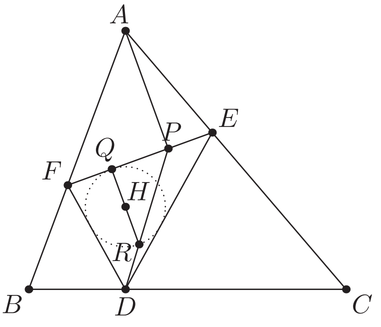
Solution 4.44



We claim the fixed point is the orthocenter H of $\triangle ABC$.

We know that $\overline{BH} \parallel \overline{XP}$. Moreover, $\overline{RP}$ bisects $\overline{XH}$ by Lemma 4.4. This is enough to deduce that $HRXP$ is a parallelogram. Hence ℓ is precisely line PH, as needed.

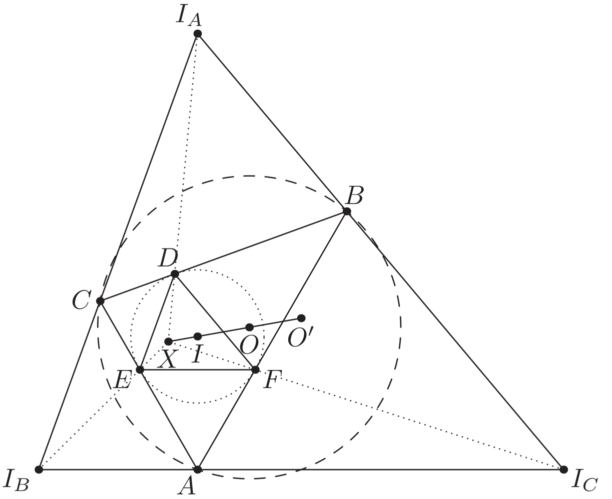
Solution 4.45



The answer is 1; we prove H is the midpoint of $\overline{QR}$. By Lemma 4.6, H is the incenter of $\triangle DEF$ and A is the D-excenter. Hence by applying Lemma 4.9 we are done.

Solution 4.50

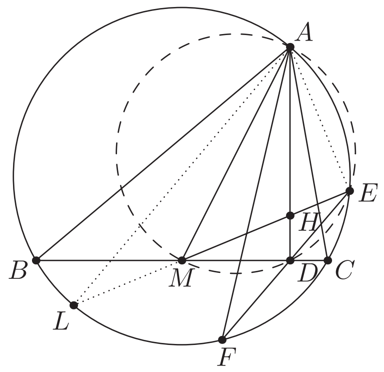
Let I_A, I_B, I_C denote the excenters. By Lemma 4.14, line A_0D is just line I_AD, and similarly for the others.



Hence there is a homothety taking $\triangle DEF$ to $\triangle I_AI_BI_C$. This implies already that lines A_0D, B_0E, C_0F concur at some point X.

Let O' be the circumcenter of triangle $I_AI_BI_C$. Because IO is the Euler line of $I_AI_BI_C$ (with nine-point center O), it passes through O'. The homothety maps the circumcircle I of $\triangle DEF$ to the circumcenter O' of $\triangle I_AI_BI_C$. It follows that X lies on $\overline{IO'}$, so we are done.

Solution 4.52



We claim that $\overline{AF}$ is a symmedian, from which everything else follows. Let L be the reflection of H over M; by Lemma 1.17, we obtain $\angle MEA = \angle LEA = 90°$. Hence $MDEA$ is cyclic.

Now, we compute

$$\measuredangle MAC + \measuredangle CAE = \measuredangle MAE = \measuredangle MDE = \measuredangle BDE$$

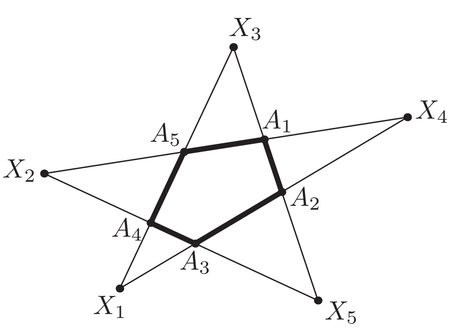
but

$$\measuredangle BDE = \measuredangle BED + \measuredangle BDE = \measuredangle BEF + \measuredangle CBE = \measuredangle BAF + \measuredangle CAE$$

hence $\measuredangle BAF = \measuredangle MAC$ as required.

C.2 Solutions to Chapters 5–7

Solution 5.16



By the law of sines on $\triangle A_i A_{i+1} X_{i+3}$, we find that

$$\frac{A_i X_{i+3}}{A_{i+1} X_{i+3}} = \frac{\sin \angle A_i A_{i+1} X_{i+3}}{\sin \angle A_{i+1} A_i X_{i+3}}.$$

But we have $\angle A_{i+1}A_iX_{i+3} = \angle A_{i-1}A_iX_{i+2}$, so in fact

$$\frac{A_iX_{i+3}}{A_{i+1}X_{i+3}} = \frac{\sin\angle A_iA_{i+1}X_{i+3}}{\sin\angle A_{i-1}A_iX_{i+2}}.$$

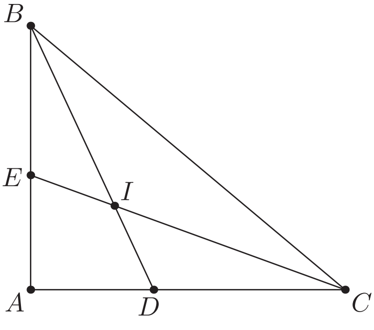
Hence we obtain

$$\prod_{i=1}^{5}\frac{A_iX_{i+3}}{A_{i+1}X_{i+3}} = \prod_{i=1}^{5}\frac{\sin\angle A_iA_{i+1}X_{i+3}}{\sin\angle A_{i-1}A_iX_{i+2}} = 1$$

which is what we wanted to prove.

Solution 5.21

The answer is no. We prove that it is not even possible that AB, AC, CI, IB are all integers.



It is easy to see, say by Example 1.4, that $\angle BIC = 135^\circ$. Thus

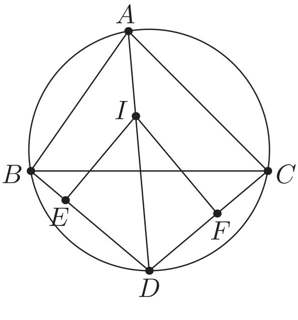
$$\begin{aligned} BC^2 &= BI^2 + CI^2 - 2BI\cdot CI\cos\angle BIC \\ &= BI^2 + CI^2 - BI\cdot CI\cdot\sqrt{2} \end{aligned}$$

by the law of cosines. Yet $BC^2 = AB^2 + AC^2$. So we derive

$$\sqrt{2} = \frac{BI^2 + CI^2 - AB^2 - AC^2}{BI\cdot CI}.$$

Since $\sqrt{2}$ is irrational, it is impossible that BI, CI, AB, AC are all integers.

Solution 5.22



Let $x = DB = DI = DC$ (again using Lemma 1.18). In that case, since $\angle IDE = \angle ADB = \angle ACB$ we have

$$IE = ID \cdot \sin \angle IDE = x \sin C = x \cdot \frac{c}{2R}.$$

Similarly, $IF = x \cdot \frac{b}{2R}$. On the other hand, $AD \cdot a = x \cdot (b + c)$ by Ptolemy's theorem on $ABDC$, so $AD = \frac{x(b+c)}{a}$. Putting this all together, we find that

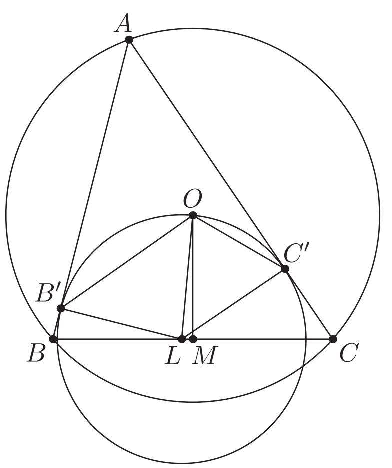
$$\frac{1}{2}\frac{x(b+c)}{a} = IE + IF = \frac{x}{2R}(b+c).$$

Consequently we find $a = R$.

Therefore, $\sin A = \frac{a}{2R} = \frac{1}{2}$ is necessary and sufficient. So the acceptable values are $\angle A = 30^\circ$ and $\angle A = 150^\circ$.

Solution 5.27

Let M be the midpoint of $\overline{BC}$.



First, we are going to prove that $\angle A < 60^\circ$. Let $\alpha = \angle A$. Then

$$\angle BOC = 2\angle BAC = 2\alpha.$$

Also,

$$\begin{aligned}\angle B'OC' &= \frac{1}{2}\left(360^\circ - \angle B'LC'\right)\\ &= 180^\circ - \frac{1}{2}\left(180^\circ - \angle B'AC'\right)\\ &= 90^\circ + \frac{1}{2}\alpha.\end{aligned}$$

We know $\angle B'OC' > \angle BOC$; therefore $90^\circ + \frac{1}{2}\alpha > 2\alpha$, which implies $\alpha < 60^\circ$ as needed.

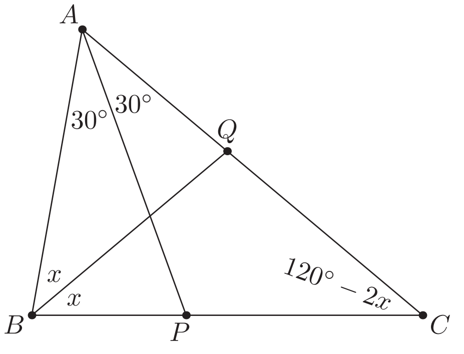
Now for the finish. It suffices to prove that $OL > \frac{1}{2}R$, where R is the circumradius of ABC. But

$$OL \geq OM = R \cdot \cos(\alpha) > R\cos(60^\circ) = \frac{1}{2}R$$

and we are done.

Solution 5.29

The answer is $\angle B = 80^\circ$ and $\angle C = 40^\circ$. Set $x = \angle ABQ = \angle QBC$, so that $\angle QCB = 120^\circ - 2x$. We observe $\angle AQB = 120^\circ - x$ and $\angle APB = 150^\circ - 2x$.



Now by the law of sines, we may compute

$$\begin{aligned} BP &= AB \cdot \frac{\sin 30^\circ}{\sin(150^\circ - 2x)} \\ AQ &= AB \cdot \frac{\sin x}{\sin(120^\circ - x)} \\ QB &= AB \cdot \frac{\sin 60^\circ}{\sin(120^\circ - x)}. \end{aligned}$$

So, the relation $AB + BP = AQ + QB$ is exactly

$$1 + \frac{\sin 30^\circ}{\sin(150^\circ - 2x)} = \frac{\sin x + \sin 60^\circ}{\sin(120^\circ - x)}.$$

At this point, we have completely transformed our geometry problem into a direct algebra equation, hardly worthy of its place as Problem 5 at the IMO. Many solutions are possible at this point, and we present only one of them.

First of all, we can write

$$\sin x + \sin 60^\circ = 2\sin\left(\frac{1}{2}(x + 60^\circ)\right)\cos\left(\frac{1}{2}(x - 60^\circ)\right).$$

On the other hand, $\sin(120^\circ - x) = \sin(x + 60^\circ)$ and

$$\sin(x + 60^\circ) = 2\sin\left(\frac{1}{2}(x + 60^\circ)\right)\cos\left(\frac{1}{2}(x + 60^\circ)\right)$$

so

$$\frac{\sin x + \sin 60^\circ}{\sin(120^\circ - x)} = \frac{\cos\left(\frac{1}{2}x - 30^\circ\right)}{\cos\left(\frac{1}{2}x + 30^\circ\right)}.$$

Let $y = \frac{1}{2}x$ for brevity now. Then

$$\begin{aligned}\frac{\cos(y - 30^\circ)}{\cos(y + 30^\circ)} - 1 &= \frac{\cos(y - 30^\circ) - \cos(y + 30^\circ)}{\cos(y + 30^\circ)}\\ &= \frac{2\sin(30^\circ)\sin y}{\cos(y + 30^\circ)}\\ &= \frac{\sin y}{\cos(y + 30^\circ)}.\end{aligned}$$

Hence the problem is just

$$\frac{\sin 30^\circ}{\sin(150^\circ - 4y)} = \frac{\sin y}{\cos(y + 30^\circ)}.$$

Equivalently,

$$\begin{aligned}\cos(y + 30^\circ) &= 2\sin y \sin(150^\circ - 4y)\\ &= \cos(5y - 150^\circ) - \cos(150^\circ - 3y)\\ &= -\cos(5y + 30^\circ) + \cos(3y + 30^\circ).\end{aligned}$$

Now we are home free, because $3y + 30^\circ$ is the average of $y + 30^\circ$ and $5y + 30^\circ$. That means we can write

$$\frac{\cos(y + 30^\circ) + \cos(5y + 30^\circ)}{2} = \cos(3y + 30^\circ)\cos(2y).$$

Hence

$$\cos(3y + 30^\circ)\,(2\cos(2y) - 1) = 0.$$

Recall that

$$y = \frac{1}{2}x = \frac{1}{4}\angle B < \frac{1}{4}(180^\circ - \angle A) = 30^\circ.$$

Hence it is not possible that $\cos(2y) = \frac{1}{2}$, since the smallest positive value of y that satisfies this is $y = 30^\circ$. So $\cos(3y + 30^\circ) = 0$. The only permissible value of y is then $y = 20^\circ$, giving $\angle B = 80^\circ$ and $\angle C = 40^\circ$.

Solution 5.30

The problem condition is equivalent to

$$ac + bd = (b + d)^2 - (a - c)^2$$

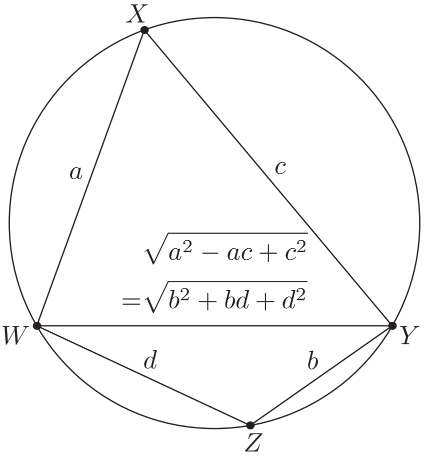
or

$$a^2 - ac + c^2 = b^2 + bd + d^2.$$

Let us construct a quadrilateral $WXYZ$ such that $WX = a$, $XY = c$, $YZ = b$, $ZW = d$, and

$$WY = \sqrt{a^2 - ac + c^2} = \sqrt{b^2 + bd + d^2}.$$

Then by the law of cosines, we obtain $\angle WXY = 60^\circ$ and $\angle WZY = 120^\circ$. Hence this quadrilateral is cyclic.



By Theorem 5.10, we find that

$$WY^2 = \frac{(ab + cd)(ad + bc)}{ac + bd}.$$

Now assume for contradiction that that $ab + cd$ is a prime p. Recall that we assumed $a > b > c > d$. It follows, e.g. by the so-called rearrangement inequality, that

$$p = ab + cd > ac + bd > ad + bc.$$

Let $y = ac + bd$ and $x = ad + bc$ now. The point is that

$$p \cdot \frac{x}{y}$$

can never be an integer if p is prime and $x < y < p$ (why?). But $WY^2 = a^2 - ac + c^2$ is clearly an integer, and this is a contradiction.

Hence $ab + cd$ cannot be prime.

Solution 6.30

We have that P lies on $\overline{AB}$ if and only if

$$\frac{p - a}{p - b} = \overline{\left(\frac{p - a}{p - b}\right)}.$$

Because $\overline{a} = \frac{1}{a}$ and $\overline{b} = \frac{1}{b}$, the right-hand side equals

$$\frac{\overline{p} - \overline{a}}{\overline{p} - \overline{b}} = \frac{\overline{p} - \frac{1}{a}}{\overline{p} - \frac{1}{b}}.$$

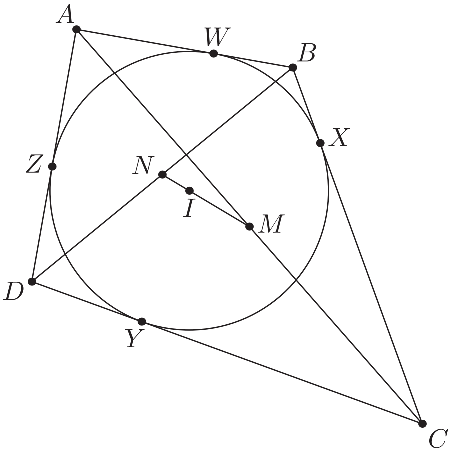
Clearing the denominators, we find that the condition is equivalent to

$$\begin{aligned} 0 &= (p-a)\left(\overline{p}-\frac{1}{b}\right)-(p-b)\left(\overline{p}-\frac{1}{a}\right) \\ &= (b-a)\overline{p}-\left(\frac{1}{b}-\frac{1}{a}\right)p+\frac{a}{b}-\frac{b}{a} \\ &= (b-a)\overline{p}-\frac{a-b}{ab}p+\frac{a^2-b^2}{ab} \\ &= \frac{b-a}{ab}\left(ab\overline{p}+p-(a+b)\right). \end{aligned}$$

Since $a \neq b$, we find the condition is exactly $ab\overline{p}+p-(a+b)=0$, which is what we wanted to prove.

Solution 6.32

Let W, X, Y, Z denote the tangency points of the incircle of $ABCD$ to the sides $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$. Let M be the midpoint of $\overline{AC}$ and N the midpoint of $\overline{BD}$.



We apply complex numbers with the circumcircle of $WXYZ$ as the unit circle; our free variables will be w, x, y, z. Using Lemma 6.19, we find

$$a=\frac{2zw}{z+w}, \quad b=\frac{2wx}{w+x}, \quad c=\frac{2xy}{x+y}, \quad d=\frac{2yz}{y+z}.$$

Thus

$$\begin{aligned} m &= \frac{a+c}{2} \\ &= \frac{1}{2}\left(\frac{2zw}{z+w} + \frac{2xy}{x+y}\right) \\ &= \frac{zw(x+y)+xy(z+w)}{(z+w)(x+y)} \\ &= \frac{wxy+xyz+yzw+zwx}{(z+w)(x+y)}. \end{aligned}$$

Similarly,

$$n = \frac{b+d}{2} = \frac{wxy+xyz+zyw+zwx}{(w+x)(y+z)}.$$

To show that these are collinear with the incenter I, which has coordinate 0, we only have to show that the quotient $\frac{m-0}{n-0}$ is a real number. But the quotient is just

$$\frac{m}{n} = \frac{(w+x)(y+z)}{(z+w)(x+y)}.$$

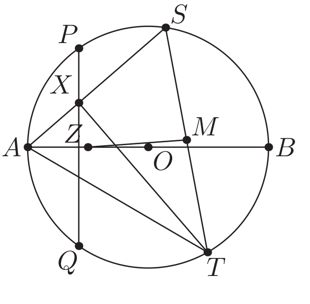
Its conjugate is

$$\overline{\left(\frac{m}{n}\right)} = \frac{\left(\frac{1}{w}+\frac{1}{x}\right)\left(\frac{1}{y}+\frac{1}{z}\right)}{\left(\frac{1}{z}+\frac{1}{w}\right)\left(\frac{1}{x}+\frac{1}{y}\right)} = \frac{\frac{w+x}{wx}\cdot\frac{y+z}{yz}}{\frac{z+w}{zw}\cdot\frac{x+y}{xy}} = \frac{(w+x)(y+z)}{(z+w)(x+y)}.$$

Hence $\frac{m}{n}$ is equal to its conjugate, so it is real. Therefore we are done.

Solution 6.35

Toss on the complex unit circle with $a=-1$, $b=1$, $z=-\frac{1}{2}$. Let s and t be on the unit circle. We claim Z is the center.



By, Lemma 6.11

$$x = \frac{1}{2}(s+t-1+s/t).$$

Then

$$4\operatorname{Re} x + 2 = s+t+\frac{1}{s}+\frac{1}{t}+\frac{s}{t}+\frac{t}{s}$$

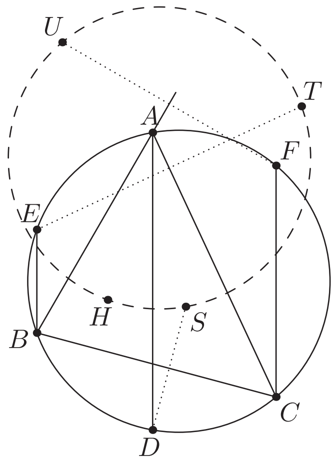
depends only on P and Q, and not on X. But

$$4\left|z-\frac{s+t}{2}\right|^2 = |s+t+1|^2 = 3 + (4\operatorname{Re} x + 2)$$

which implies that $\frac{1}{2}(s+t)$ has a fixed distance from z, as desired.

Solution 6.36

We of course set (ABC) as the unit circle, but moreover, by a suitable rotation we let $\overline{AD}$, $\overline{BE}$, $\overline{CF}$ lie perpendicular to the real axis. This will cause $d = \overline{a}$ and so on.



By Lemma 6.11, it is easy to see that

$$s = b + c - bc\overline{d} = b + c - abc.$$

Similarly,

$$t = c + a - abc \quad \text{and} \quad u = a + b - abc.$$

We now wish to apply Theorem 6.15 to deduce the points S, T, U, H are concyclic. Compute

$$\frac{u-h}{t-h} : \frac{u-s}{t-s} = \frac{-c-abc}{-b-abc} : \frac{a-c}{a-b} = \frac{c(a-b)(ab-1)}{b(a-c)(ac-1)}.$$

We are done once we check that this expression is a real number. The conjugate of this expression is

$$\begin{aligned}
\frac{\frac{1}{c}\left(\frac{1}{a}-\frac{1}{b}\right)\left(\frac{1}{ab}-1\right)}{\frac{1}{b}\left(\frac{1}{a}-\frac{1}{c}\right)\left(\frac{1}{ac}-1\right)} &= \frac{\frac{1}{c}\cdot\frac{b-a}{ab}\cdot\frac{1-ab}{ab}}{\frac{1}{b}\cdot\frac{c-a}{ac}\cdot\frac{1-ac}{ac}} \\
&= \frac{c(b-a)(1-ab)}{b(c-a)(1-ac)} \\
&= \frac{c(a-b)(ab-1)}{b(a-c)(ac-1)}
\end{aligned}$$

as needed.

Solution 6.38

We apply complex numbers with (ABC) the unit circle. Observe that $x + y = 0$ and $xy + bc = 0$ (one way to see the latter expression is by Example 6.10). Moreover, the condition $\triangle DPO \sim \triangle PEO$ is just

$$\frac{d-p}{p-0} = \frac{p-e}{e-0} \Leftrightarrow p^2 - pe = de - pe \Leftrightarrow p^2 = de.$$

Now we can compute

$$\begin{aligned}
(PX \cdot PY)^2 &= |p-x|^2\,|p-y|^2 \\
&= (p-x)(\overline{p}-\overline{x})(p-y)(\overline{p}-\overline{y}) \\
&= \left(p^2 - (x+y)p + xy\right)\left(\overline{p}^2 - (\overline{x}+\overline{y})\,\overline{p} + \overline{xy}\right) \\
&= (p^2 + xy)\left(\overline{p}^2 + \overline{xy}\right) \\
&= (de - bc)\left(\overline{de} - \overline{bc}\right) \\
&= |de - bc|^2 .
\end{aligned}$$

Thus $PX \cdot PY = |de - bc|$. Now we can also compute, using Lemma 6.11, that $d = a + c - \frac{ac}{b}$ and $e = a + b - \frac{ab}{c}$. Therefore,

$$\begin{aligned}
de &= \left(a + c - \frac{ac}{b}\right)\left(a + b - \frac{ab}{c}\right) \\
&= a^2 + ab + ac + bc - \frac{a^2c}{b} - ac - \frac{a^2b}{c} - ab + a^2 \\
&= 2a^2 - \frac{a^2c}{b} - \frac{a^2b}{c} + bc.
\end{aligned}$$

Hence

$$\begin{aligned}
PX \cdot PY &= |de - bc| \\
&= \left|2a^2 - \frac{a^2c}{b} - \frac{a^2b}{c}\right| \\
&= \left|-\frac{a^2}{bc}(b-c)^2\right| \\
&= \left|-\frac{a^2}{bc}\right| |b-c|^2 \\
&= BC^2.
\end{aligned}$$

From $\tan A = \frac{3}{4}$ we can derive $\cos A = \frac{4}{5}$, so the law of cosines gives

$$BC^2 = 13^2 + 25^2 - 2 \cdot 13 \cdot 25 \cdot \frac{4}{5} = 274$$

which is the final answer.

Solution 6.39

First, observe that in general, if $z = a + bi$, then $\tan(\arg z) = \frac{b}{a}$, with the quantity being undefined when $a = 0$. This just follows from the geometric interpretation of complex numbers.

Let $\alpha = 1 + xi$, $\beta = 1 + yi$, $\gamma = 1 + zi$. Then $\arg\alpha = A$, $\arg\beta = B$, $\arg\gamma = C$. Thus $\arg(\alpha\beta\gamma)$ equals $A + B + C$ (again all arguments are taken modulo 360°). But you can check that

$$\begin{aligned}\alpha\beta\gamma &= 1 + (x + y + z)i + (xy + yz + zx)i^2 + xyzi^3 \\ &= (1 - (xy + yz + zx)) + (x + y + z - xyz)\, i.\end{aligned}$$

Hence

$$\frac{x + y + z - xyz}{1 - (xy + yz + zx)} = \tan\arg(\alpha\beta\gamma) = \tan(A + B + C)$$

as required.

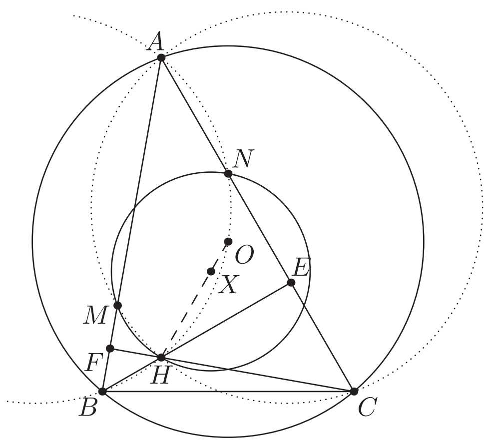
By generalizing to multiple variables and repeating the same calculation, one can obtain the following: given $x_i = \tan\theta_i$ for $i = 1, 2, \ldots, n$, we have

$$\tan(\theta_1 + \cdots + \theta_n) = \frac{e_1 - e_3 + e_5 - e_7 + \ldots}{1 - e_2 + e_4 - e_6 + \ldots}$$

where e_m is the sum of the $\binom{n}{m}$ possible products of m of the x_i. The above result was the special case $n = 3$.

Solution 6.42

Let $\overline{BE}$ and $\overline{CF}$ be altitudes of $\triangle ABC$.



First, we claim that M is the reflection of B over F. Indeed, we have that

$$\measuredangle BMH = \measuredangle AMH = \measuredangle ACH = \measuredangle ECF = \measuredangle EBF = \measuredangle HBM$$

implying that $\triangle MHB$ is isosceles. As $\overline{HF} \perp \overline{MB}$, the conclusion follows. Similarly, we can see that N is the reflection of C over E.

Now we can apply complex numbers with (ABC) as the unit circle. Hence we have $f = \frac{1}{2}(a+b+c-ab\overline{c})$ (via Lemma 6.11), and hence

$$m = 2f - b = a + c - ab\overline{c}.$$

Similarly,

$$n = a + b - ac\overline{b}.$$

Now we wish to compute the circumcenter X of $\triangle HMN$, where $h = a+b+c$. Let M' be the point corresponding to $m - h = -b - ab\overline{c}$ and N' be the point corresponding to $n - h = -c - ac\overline{b}$, noting that O corresponds to $h - h = 0$. Then the circumcenter of $\triangle M'N'O$ corresponds to the point $x - h$. But we can compute the circumcenter of $\triangle M'N'O$ using Lemma 6.24; it is

$$\begin{aligned}
x - h &= \frac{(m-h)(n-h)\left(\overline{(m-h)} - \overline{(n-h)}\right)}{\overline{(m-h)}(n-h) - (m-h)\overline{(n-h)}} \\
&= \frac{\left(-b - \frac{ab}{c}\right)\left(-c - \frac{ac}{b}\right)\left(\left(-\frac{1}{b} - \frac{c}{ab}\right) - \left(-\frac{1}{c} - \frac{b}{ac}\right)\right)}{\left(-\frac{1}{b} - \frac{c}{ab}\right)\left(-c - \frac{ac}{b}\right) - \left(-b - \frac{ab}{c}\right)\left(-\frac{1}{c} - \frac{b}{ac}\right)} \\
&= \frac{\left(b + \frac{ab}{c}\right)\left(c + \frac{ac}{b}\right)\left(\left(\frac{1}{b} + \frac{c}{ab}\right) - \left(\frac{1}{c} + \frac{b}{ac}\right)\right)}{\left(\frac{1}{b} + \frac{c}{ab}\right)\left(c + \frac{ac}{b}\right) - \left(b + \frac{ab}{c}\right)\left(\frac{1}{c} + \frac{b}{ac}\right)}.
\end{aligned}$$

Multiplying the numerator and denominator by ab^2c^2,

$$\begin{aligned}
x - h &= \frac{bc\,(a+b)\,(a+c)\,(c(a+c) - b(a+b))}{c^3(a+b)(a+c) - b^3(a+b)(a+c)} \\
&= \frac{bc\left(c^2 - b^2 + a(c-b)\right)}{c^3 - b^3} \\
&= \frac{bc(c-b)(a+b+c)}{(c-b)(b^2+bc+c^2)} \\
&= \frac{bc(a+b+c)}{b^2+bc+c^2}.
\end{aligned}$$

So

$$x = h + \frac{bc(a+b+c)}{b^2+bc+c^2} = h\left[1 + \frac{bc}{b^2+bc+c^2}\right].$$

Finally, to show X, H, O are collinear, we only need to prove $\frac{x}{h} = \frac{bc}{b^2+bc+c^2} + 1$ is real. It is equivalent to show $\frac{bc}{b^2+bc+c^2}$ is real, but its conjugate is

$$\overline{\left(\frac{bc}{b^2+bc+c^2}\right)} = \frac{\frac{1}{bc}}{\frac{1}{b^2} + \frac{1}{bc} + \frac{1}{c^2}} = \frac{bc}{b^2+bc+c^2}$$

and the proof is complete.

Solution 6.44

We apply complex numbers with $(ABCD)$ as the unit circle. The problem is equivalent to proving that

$$\frac{\frac{1}{2}p - \frac{1}{2}(o_1 + o_3)}{\frac{1}{2}\overline{p} - \frac{1}{2}(\overline{o_1} + \overline{o_3})} = \frac{\frac{1}{2}p - \frac{1}{2}(o_2 + o_4)}{\frac{1}{2}\overline{p} - \frac{1}{2}(\overline{o_2} + \overline{o_4})}.$$

First, we compute

$$\begin{aligned} o_1 &= \begin{vmatrix} a & a\overline{a} & 1 \\ b & b\overline{b} & 1 \\ p & p\overline{p} & 1 \end{vmatrix} \div \begin{vmatrix} a & \overline{a} & 1 \\ b & \overline{b} & 1 \\ p & \overline{p} & 1 \end{vmatrix} \\ &= \begin{vmatrix} a & 1 & 1 \\ b & 1 & 1 \\ p & p\overline{p} & 1 \end{vmatrix} \div \begin{vmatrix} a & \frac{1}{a} & 1 \\ b & \frac{1}{b} & 1 \\ p & \overline{p} & 1 \end{vmatrix} \\ &= \begin{vmatrix} a & 0 & 1 \\ b & 0 & 1 \\ p & p\overline{p} - 1 & 1 \end{vmatrix} \div \begin{vmatrix} a & \frac{1}{a} & 1 \\ b & \frac{1}{b} & 1 \\ p & \overline{p} & 1 \end{vmatrix} \\ &= \frac{(p\overline{p} - 1)(b - a)}{\frac{a}{b} - \frac{b}{a} + p(\frac{1}{a} - \frac{1}{b}) + \overline{p}(b - a)} \\ &= \frac{p\overline{p} - 1}{\frac{p}{ab} + \overline{p} - \frac{a+b}{ab}}. \end{aligned}$$

The conjugate of this expression is easier to work with; we have

$$\overline{o_1} = \frac{p\overline{p} - 1}{ab\overline{p} + p - (a + b)}.$$

Similarly,

$$\overline{o_3} = \frac{p\overline{p} - 1}{cd\overline{p} + p - (c + d)}.$$

In what follows, we let $s_1 = a + b + c + d$, $s_2 = ab + bc + cd + da + ac + bd$, $s_3 = abc + bcd + cda + dab$, and $s_4 = abcd$ for brevity. Then,

$$\begin{aligned} &\overline{o_1} + \overline{o_3} - \overline{p} \\ &= (p\overline{p} - 1)\left(\frac{1}{ab\overline{p} + p - (a + b)} + \frac{1}{cd\overline{p} + p - (c + d)}\right) - \overline{p} \\ &= \frac{(p\overline{p} - 1)(2p + (ab + cd)\overline{p} - s_1)}{(ab\overline{p} + p - (a + b))(cd\overline{p} + p - (c + d))} - \overline{p}. \end{aligned}$$

Consider the fraction in the above expansion. One can check that the denominator expands as

$$\mathcal{D} = s_4\overline{p}^2 + (ab + cd)\,p\overline{p} + p^2 - s_3\overline{p} - s_1 p + (ac + ad + bc + bd).$$

On the other hand, the numerator is equal to

$$\mathcal{N} = (2p - s_1)(p\overline{p} - 1) + (ab + cd)\overline{p}(p\overline{p} - 1).$$

Thus,

$$\overline{o_1} + \overline{o_3} - \overline{p} = \frac{\mathcal{N} - \overline{p}\mathcal{D}}{\mathcal{D}}.$$

We claim that the expression $\mathcal{N} - \overline{p}\mathcal{D}$ is symmetric in a, b, c, d. To see this, we need only look at the terms of $\mathcal{N}$ and $\mathcal{D}$ that are not symmetric in a, b, c, d. These are $(ab+cd)\overline{p}(p\overline{p}-1)$ and $(ab+cd)p\overline{p} + (ac+ad+bd+bc)$, respectively. Subtracting $\overline{p}$ times the latter from the former yields $-s_2\overline{p}$. Hence $\mathcal{N} - \overline{p}\mathcal{D}$ is symmetric in a, b, c, d, as claimed.* Now we may set $\mathcal{S} = \mathcal{N} - \overline{p}\mathcal{D}$.

Thus

$$\begin{aligned}
\frac{o_1 + o_3 - p}{\overline{o_1} + \overline{o_3} - \overline{p}} &= \frac{\overline{\mathcal{S}}/\overline{\mathcal{D}}}{\mathcal{S}/\mathcal{D}} \\
&= \frac{\overline{\mathcal{S}}}{\mathcal{S}} \cdot \frac{\mathcal{D}}{\overline{\mathcal{D}}} \\
&= \frac{\overline{\mathcal{S}}}{\mathcal{S}} \cdot \frac{(ab\overline{p} + p - (a+b))(cd\overline{p} + p - (c+d))}{(\frac{1}{ab}p + \overline{p} - \frac{1}{a} - \frac{1}{b})(\frac{1}{cd}p + \overline{p} - \frac{1}{c} - \frac{1}{d})} \\
&= \frac{\overline{\mathcal{S}}}{\mathcal{S}} \cdot abcd.
\end{aligned}$$

Hence, we deduce

$$\frac{o_1 + o_3 - p}{\overline{o_1} + \overline{o_3} - \overline{p}}$$

is in fact symmetric in a, b, c, d. Hence if we repeat the same calculation with $\frac{o_2+o_4-p}{\overline{o_2}+\overline{o_4}-\overline{p}}$, we must obtain exactly the same result. This completes the solution.

Solution 6.45

We use complex numbers, since the condition in its given form is an abomination. Let a denote the number in the complex plane corresponding to A, et cetera, and consider the quantity

$$\frac{b-a}{f-a} \cdot \frac{d-c}{b-c} \cdot \frac{f-e}{d-e}.$$

By the first condition, the argument of this complex number is 360°, which means it is a positive real. However, the second condition implies that it has norm 1. We deduce that it is actually equal to 1.

So, we are given that

$$0 = (a-b)(c-d)(e-f) + (b-c)(d-e)(f-a)$$

* In fact, if you really want to do the computation you can check that $\mathcal{N} - \overline{p}\mathcal{D} = -s_4\overline{p}^3 + p^2\overline{p} + s_3\overline{p}^2 - s_2\overline{p} + \overline{p} + 2p + s - 1$. But we will not need to do anything with this expression other than notice that it is symmetric.

and wish to show that

$$|(a-b)(c-e)(d-f)| = |(d-e)(f-b)(a-c)|.$$

But now observe that

$$\begin{aligned}
&[(a-b)(c-d)(e-f)+(b-c)(d-e)(f-a)] \\
&-[(a-b)(c-e)(d-f)+(d-e)(f-b)(a-c)] \\
&=((c-d)(e-f)-(c-e)(d-f))(a-b) \\
&+((b-c)(f-a)-(f-b)(a-c))(d-e) \\
&=(f-c)(d-e)(a-b)+(f-c)(b-a)(d-e) \\
&=0.
\end{aligned}$$

So in fact $(a-b)(c-e)(d-f) = -(d-e)(f-b)(a-c)$ and the result is obvious.

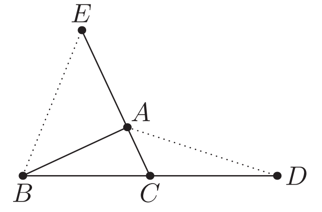
Solution 7.33

It is easy to see by similar triangles that we have $PB = c^2/a$. Hence, $P = \left(0, 1-\frac{c^2}{a^2}, \frac{c^2}{a^2}\right)$. Therefore, we derive

$$M = \left(-1, 2-\frac{2c^2}{a^2}, \frac{2c^2}{a^2}\right) = \left(-a^2 : 2a^2-2c^2 : 2c^2\right).$$

Similarly, $N = \left(-a^2 : 2b^2 : 2a^2-2b^2\right)$. Therefore, $\overline{BM}$ and $\overline{CN}$ meet at $(-a^2 : 2b^2 : 2c^2)$ which clearly lies on the circumcircle.

Solution 7.34



It is easy to compute $D = (0, -1, 2)$ and $E = (3, 0, -2)$. Hence

$$\overrightarrow{AD} = (-1, -1, 2) \quad \text{and} \quad \overrightarrow{BE} = (3, -1, -2).$$

Applying the distance formula, the condition $AD = BE$ become

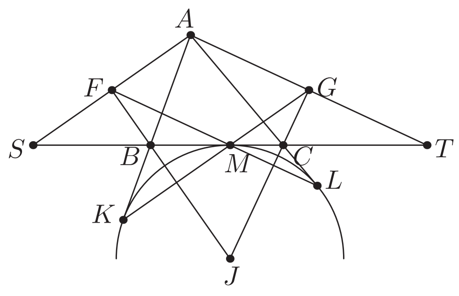
$$\begin{aligned}
&-a^2(-1)(2) - b^2(2)(-1) - c^2(-1)(-1) \\
&= -a^2(-1)(-2) - b^2(-2)(3) - c^2(3)(-1)
\end{aligned}$$

which is

$$2a^2+2b^2-c^2 = -2a^2+6b^2+3c^2.$$

Rearranging gives $a^2 = b^2+c^2$, as needed.

Solution 7.36



As usual we use reference triangle ABC, and remind the reader that $s = \frac{1}{2}(a+b+c)$.

Since $AK = s$ gives $BK = s - c$, we have $K = (-(s-c) : s : 0)$. Also, $J = (-a : b : c)$ and $M = (0 : s-b : s-c)$. The point G lies on $\overline{CJ}$, so we put $G = (-a : b : t)$ and compute the determinant indicating that G, M, K are collinear, namely

$$0 = \begin{vmatrix} -a & b & t \\ 0 & s-b & s-c \\ c-s & s & 0 \end{vmatrix}.$$

Expanding the determinant yields

$$0 = -a(-s(s-c)) - (s-c)(b(s-c) - t(s-b))$$

from which it follows that $t = \frac{b(s-c)-as}{s-b}$. Consequently,

$$G = (-a(s-b) : b(s-b) : b(s-c) - as).$$

So

$$T = (0 : b(s-b) : b(s-c) - as).$$

But $b(s-b) + b(s-c) - as = ba - as = -a(s-b)$, so we realize that

$$T = \left(0, -\frac{b}{a}, 1 + \frac{b}{a}\right).$$

Hence $CT = b$.

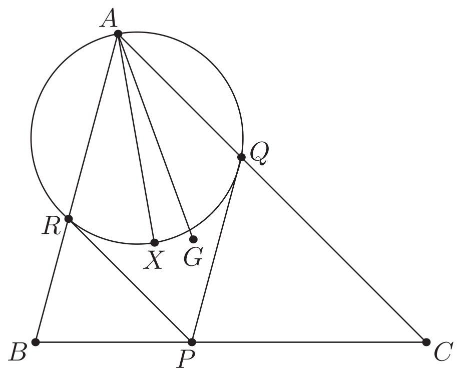
Similarly, $BS = c$. From here it is trivial to check that $MT = MS$.

Solution 7.38

Let $P = (0, s, t)$ where $s + t = 1$. One can check that $Q = (s, 0, t)$. Indeed, the normalized z-coordinates must coincide since $[AQB] = [APB]$. Similarly, $R = (t, s, 0)$. So the circumcircle of $\triangle AQR$ is given by

$$-a^2yz - b^2zx - c^2xy + (x+y+z)(ux+vy+wz) = 0$$

where u, v, w are some real numbers. Plugging in the point A gives $u = 0$. Plugging in the point Q gives $wt = b^2st$, so $w = b^2s$. Plugging in the point R gives $vs = c^2st$, so $v = c^2t$.



Thus the circumcircle has equation

$$-a^2yz - b^2zx - c^2xy + (x+y+z)\left(c^2ty + b^2sz\right) = 0.$$

Now let us consider the intersection of the A-symmedian with this circumcircle. Let the intersection be $X = (k : b^2 : c^2)$. We aim to show the value of k does not depend on s or t. But this is obvious, as substitution gives

$$-a^2b^2c^2 - 2b^2c^2k + (k+b^2+c^2)(b^2c^2)(s+t) = 0.$$

Since $s + t = 1$ and the equation is linear in k, we have exactly one solution for k. The proof ends here; there is no need to compute the value of k explicitly. (For the curious, the actual value of k is $k = -a^2 + b^2 + c^2$.)

Solution 7.42

Let X_A be the contact point of the A-excircle with $\overline{BC}$. Then $X_A = (0 : s-b : s-c)$ and Lemma 4.40 implies that $\overline{AX_A}$ and $\overline{AT_A}$ are isogonal. Since $\overline{AX_A}$, $\overline{BX_B}$, $\overline{CX_C}$ concur at the Nagel point $(s-a : s-b : s-c)$, the cevians $\overline{AT_A}$, $\overline{BT_B}$, $\overline{CT_C}$ concur at the isogonal conjugate of the Nagel point with coordinates $\left(\frac{a^2}{s-a} : \frac{b^2}{s-b} : \frac{c^2}{s-c}\right)$.

We wish to show that this point lies on line IO. Using $I = (a : b : c)$ and $O = (a^2S_A : b^2S_B : c^2S_C)$ it is equivalent to show that

$$0 = \begin{vmatrix} \frac{a^2}{s-a} & \frac{b^2}{s-b} & \frac{c^2}{s-c} \\ a^2S_A & b^2S_B & c^2S_C \\ a & b & c \end{vmatrix}.$$

Directly expanding this looks quite painful. Instead, we can factor it as

$$\frac{(abc)^2}{K^2/s} \begin{vmatrix} (s-b)(s-c) & (s-c)(s-a) & (s-a)(s-b) \\ S_A & S_B & S_C \\ \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \end{vmatrix}$$

or

$$\frac{abc}{16K^2/s} \begin{vmatrix} 4(s-b)(s-c) & 4(s-c)(s-a) & 4(s-a)(s-b) \\ 2S_A & 2S_B & 2S_C \\ 2bc & 2ca & 2ab \end{vmatrix}$$

where K^2/s abbreviates $(s-a)(s-b)(s-c)$. Now

$$4(s-b)(s-c) = a^2 - (b-c)^2 = a^2 + 2bc - b^2 - c^2 = 2S_A + 2bc.$$

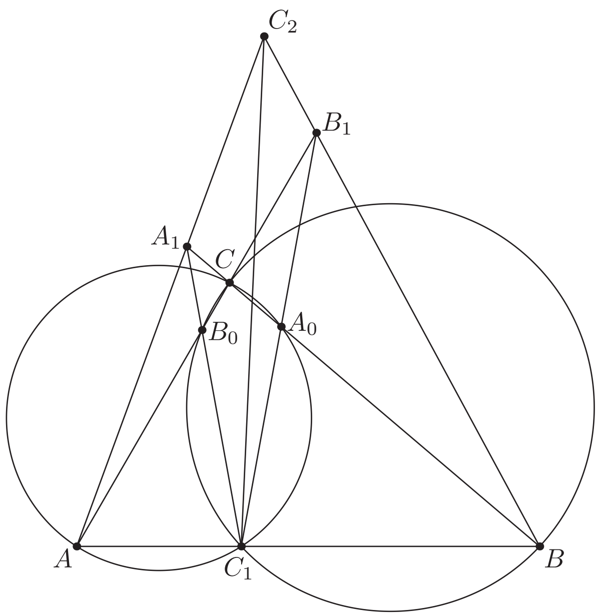
So it immediately follows that the determinant is zero (as the first row is the sum of the other two) and we are done.

Solution 7.44

We use barycentric coordinates. Let $A = (1, 0, 0)$, $B = (0, 1, 0)$, and $C = (0, 0, 1)$. Denote $a = BC$, $b = CA$, and $c = AB$. We claim that the common point is

$$K = \left(a^2 - b^2 + c^2 : b^2 - a^2 + c^2 : -c^2\right).$$

Let $C_1 = (u : v : 0)$. Let A_0 be the intersection of $\overline{C_1B_1}$ and $\overline{BC}$, and observe that AC_1A_0C is cyclic. Define B_0 analogously.



By power of a point, we observe that $BA_0 = \frac{uc}{a}$. Therefore, we obtain that

$$A_0 = \left(0 : a - \frac{uc}{a} : uc\right) = \left(0 : a^2 - uc : uc\right).$$

Combining with $C_1 = (u : v : 0)$ we therefore observe that

$$B_1 = \overline{AC} \cap \overline{C_1A_0} = (a^2 - uc : 0 : -vc).$$

Similarly,

$$A_1 = \left(0 : b^2 - vc : -uc\right).$$

Therefore,

$$C_2 = \left(u(a^2 - uc) : v(b^2 - vc) : -uvc\right).$$

Now we show that C_1, C_2, and K are collinear. Expand

$$\begin{aligned}
&\frac{-1}{c} \cdot \begin{vmatrix} u(a^2 - uc) & v(b^2 - vc) & -uvc \\ u & v & 0 \\ a^2 - b^2 + c^2 & b^2 - a^2 + c^2 & -c^2 \end{vmatrix} \\
&= \begin{vmatrix} a^2 - uc & b^2 - vc & 1 \\ 1 & 1 & 0 \\ v(a^2 - b^2 + c^2) & u(b^2 - a^2 + c^2) & c \end{vmatrix} \\
&= \left(u(b^2 - a^2 + c^2) - v(a^2 - b^2 + c^2)\right) \\
&\qquad + c\left((a^2 - uc) - (b^2 - vc)\right) \\
&= (u + v)(b^2 - a^2) + (u - v)c^2 \\
&\qquad + c(a^2 - b^2) - (u - v)c^2 \\
&= 0
\end{aligned}$$

which implies that C_1, C_2, and K are collinear, as desired.

Solution 7.47

Let ω_i be the circle with center O_i and radius r_i. Set $A_1 = (1, 0, 0)$, $A_2 = (0, 1, 0)$, $A_3 = (0, 0, 1)$, and as usual let $a = A_2A_3$ and so on. Let $A_4 = (p, q, r)$, where $p + q + r = 1$. Let $T = a^2qr + b^2rp + c^2pq$ for brevity.

The circumcircle of $\triangle A_2A_3A_4$ can be seen to have equation

$$-a^2yz - b^2zx - c^2xy + (x + y + z)\left(\frac{T}{p}x\right) = 0.$$

By Lemma 7.23, we thus have that

$$O_1A_1^2 - r_1^2 = (1 + 0 + 0) \cdot \frac{T}{p} \cdot 1 = \frac{T}{p}.$$

Similarly,

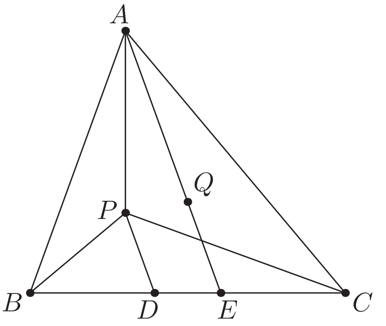
$$O_2A_2^2 - r_2^2 = \frac{T}{q} \text{ and } O_3A_3^2 - r_3^2 = \frac{T}{r}.$$

Finally, we obtain $O_4A_4^2 - r_4^2$ by plugging in A_4 into $(A_1A_2A_3)$, which gives a value of $-T$. Hence the left-hand side of our expression is

$$\frac{p}{T} + \frac{q}{T} + \frac{r}{T} - \frac{1}{T} = 0$$

since $p + q + r = 1$.

Solution 7.49



Suppose that $D = (0 : 1 : t)$ and $E = (0 : t : 1)$. Let Q be the isogonal conjugate of P; evidently Q lies on $\overline{AE}$, so $Q = (k : t : 1)$ for some k. Moreover, $P = \left(\frac{a^2}{k} : \frac{b^2}{t} : c^2\right)$. So the condition that $\overline{PD} \parallel \overline{AE}$ implies that P and D are collinear with the point at infinity $(-(1+t) : t : 1)$ along line AE, so we find

$$0 = \begin{vmatrix} a^2/k & b^2/t & c^2 \\ 0 & 1 & t \\ -(1+t) & t & 1 \end{vmatrix}$$

which can be rewritten as

$$0 = \det \begin{vmatrix} a^2/k & b^2/t & c^2 \\ 0 & 1 & t \\ -(1+t) & 1+t & 1+t \end{vmatrix} = (1+t) \begin{vmatrix} a^2/k & b^2/t & c^2 \\ 0 & 1 & t \\ -1 & 1 & 1 \end{vmatrix}.$$

Expanding the determinant, we derive that

$$0 = a^2(1-t) + k(c^2 - b^2)$$

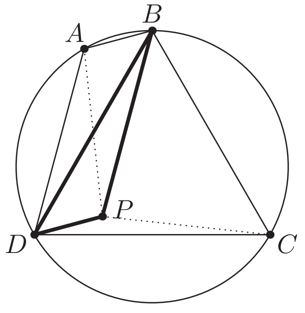
and applying Lemma 7.19 we derive that $BQ = QC$. So $\angle QBC = \angle QCB$, implying $\angle PBA = \angle PCA$.

Solution 7.52

We are going to use barycentric coordinates on $\triangle PBD$. Let $P = (1, 0, 0)$, $B = (0, 1, 0)$, $D = (0, 0, 1)$. Let $A = (au : bv : cw)$. Since C is the isogonal conjugate of A with respect to $\triangle PBD$ by the angle condition, it follows that $C = \left(\frac{a}{u} : \frac{b}{v} : \frac{c}{w}\right)$.

For brevity, we now let $S = au + bv + cw$ and $T = au^{-1} + bw^{-1} + c^{-1}$. This way, $A = \left(\frac{au}{S}, \frac{bv}{S}, \frac{cw}{S}\right)$ and $C = \left(\frac{au^{-1}}{T}, \frac{bv^{-1}}{T}, \frac{cw^{-1}}{T}\right)$. Therefore, we have

$$\overrightarrow{AP} = \left(1 - \frac{au}{S}, -\frac{bv}{S}, -\frac{cw}{S}\right) = \left(\frac{bv + cw}{S}, -\frac{bv}{S}, -\frac{cw}{S}\right)$$



and thus one can compute

$$\begin{aligned} PA^2 &= \frac{1}{S^2}\left(-a^2(bv)(cw) + b^2(cw)(bv+cw) + c^2(bv)(bv+cw)\right) \\ &= \frac{bc}{S^2}\left[-a^2vw + (bw+cv)(bv+cw)\right]. \end{aligned}$$

Performing similar calculations with C gives

$$\begin{aligned} PC^2 &= \frac{bc}{T^2}\left[-a^2(vw)^{-1}\left(bw^{-1}+cv^{-1}\right)\left(bv^{-1}+cw^{-1}\right)\right] \\ &= \frac{bc}{T^2(vw)^2}\left[-a^2vw + (bw+cv)(bv+cw)\right]. \end{aligned}$$

We would like to cancel the factor of $-a^2vw + (bw+cv)(bv+cw)$ from both sides of $PA^2 = PC^2$, but we have to check first that this factor is not zero. This follows from the fact that $PA \neq 0$ and $PC \neq 0$, since P lies in the interior of $ABCD$. Thus the division is safe, and hence $PA^2 = PC^2$ holds if and only if $S^2 = T^2(vw)^2$.

On the other hand, the quadrilateral $ABCD$ is cyclic if and only if there is some γ such that

$$-a^2yz - b^2zx - c^2xy + (x+y+z)(\gamma x) = 0$$

passes through both A and C (indeed, this is the family of circles passing through B and D). Substituting the values of $A = (au : bv : cw)$ and $C = (au^{-1} : bv^{-1} : cw^{-1})$, we see that the condition is equivalent to

$$\begin{aligned} \gamma &= \frac{-a^2(bv)(cw) - b^2(cw)(au) - c^2(au)(bv)}{au \cdot S} \\ &= \frac{-a^2(bv^{-1})(cw^{-1}) - b^2(cw^{-1})(au^{-1}) - c^2(au^{-1})(bv^{-1})}{au^{-1}T}. \end{aligned}$$

This can be rewritten as

$$-abc\frac{uvwT}{auS} = -abc \cdot \frac{(uvw)^{-1}S}{au^{-1}T}$$

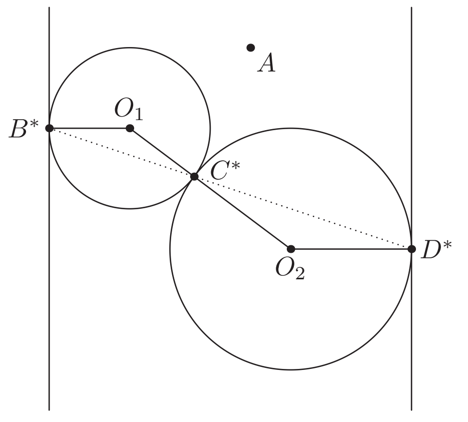
which is clearly equivalent to $S^2 = T^2(vw)^2$.

Hence $PA = PC$ if and only if $ABCD$ is cyclic.

C.3 Solutions to Chapters 8–10

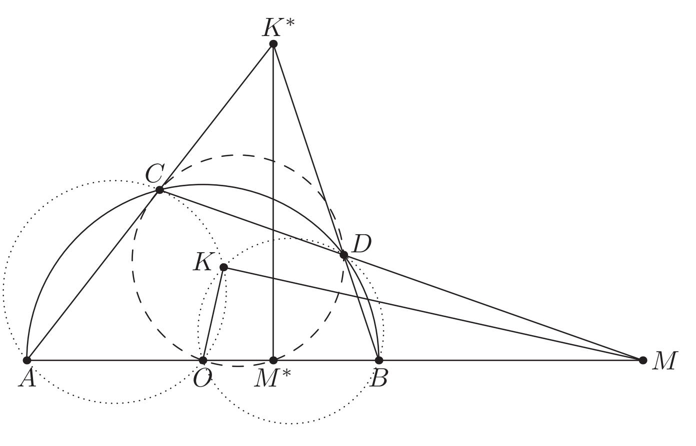
Solution 8.24

Consider an inversion around the point A. We wish to show that B^*, C^*, D^* are collinear. Our inversion gives the following image, consisting of two parallel lines and two tangent circles.



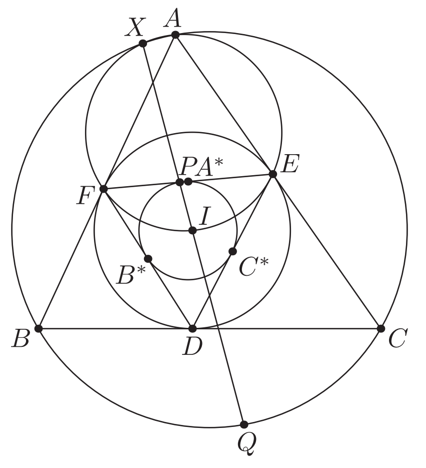
Let O_1, O_2 be the centers of the two circles in the image, such that B^* lies on the circle with center O_1 and D^* lies on the circle with center O_2. We know that O_1, C^*, O_2 are collinear. Moreover, we have $B^*O_1 = C^*O_1$ and $D^*O_2 = C^*O_2$. Finally, since $\overline{B^*O_1} \parallel \overline{D^*O_2}$ we have that $\angle B^*O_1C^* = \angle C^*O_2D^*$. Therefore, triangles $B^*O_1C^*$ and $C^*O_2D^*$ are similar. It follows that B^*, C^*, D^* are collinear, as desired.

Solution 8.27



Let us consider the inversion around the semicircle. It fixes the points A, B, C, D. Moreover, the image K^* is the intersection of lines AC and BD. Finally, the image M^* is the intersection of $\overline{AB}$ with the circumcircle of triangle OCD. We wish to prove $\angle K^*M^*O = 90^\circ$. This follows from the fact that the circumcircle of triangle OCD is in fact the nine-point circle of triangle K^*AB.

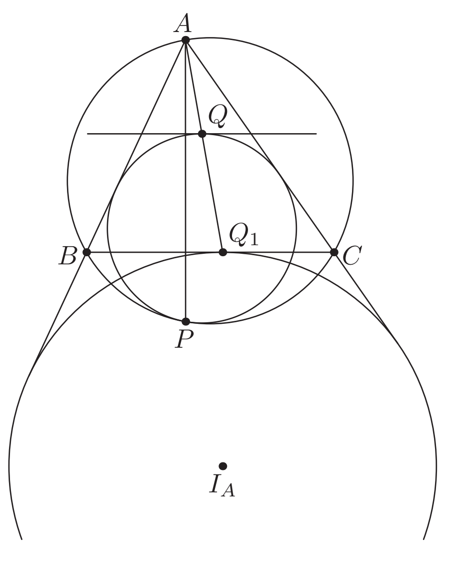
Solution 8.30



Let ray QP meet the circumcircle again at X. We have $\measuredangle IXA = \measuredangle QXA = 90^\circ$ so it follows that X lies on the circumcircle of quadrilateral $AFIE$.

Consider an inversion through the incircle. Then A^*, B^*, C^* are the midpoints of the sides of the contact triangle, and their circumcircle is the nine-point circle of triangle DEF. Moreover, since X^* lies on lines EF and XI, we derive that $P = X^*$, so P lies on the nine-point circle $(A^*B^*C^*)$ as well. Thus P is the foot of the D-altitude as required.

Solution 8.31

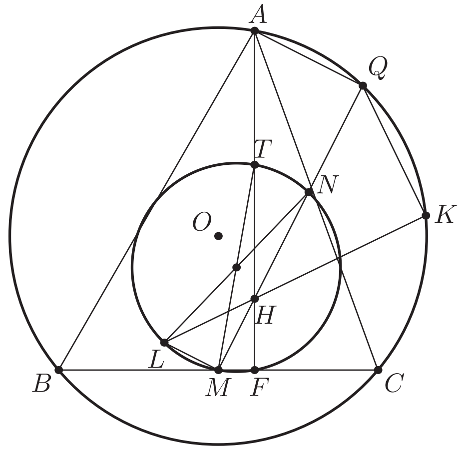


First, let us extend $\overline{AQ}$ to meet $\overline{BC}$ at Q_1. By homothety, we see that Q_1 is just the contact point of the A-excircle with $\overline{BC}$.

Now let us perform an inversion around A with radius $\sqrt{AB \cdot AC}$ followed by an reflection around the angle bisector; call this map Ψ. By Lemma 8.16, Ψ fixes B and

C. Moreover it swaps $\overline{BC}$ and (ABC). Hence, this map swaps the A-excircle with the A-mixtilinear incircle ω. Hence Ψ swaps P and Q_1. It follows that $\overline{AP}$ and $\overline{AQ_1}$ are isogonal with respect to $\angle BAC$, meaning $\angle BAP = \angle CAQ_1$. Since $\angle CAQ = \angle CAQ_1$ we are done.

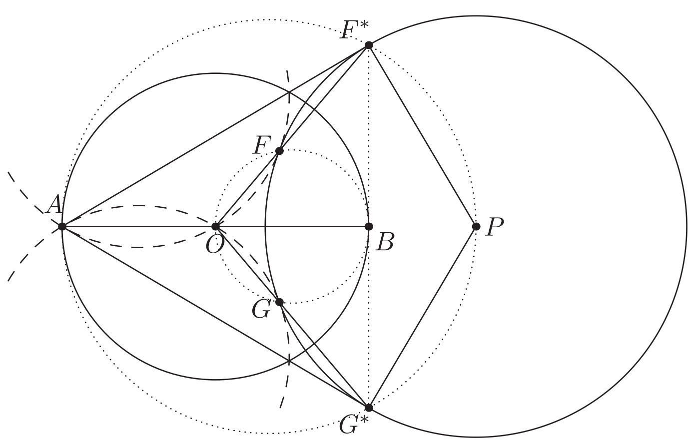
Solution 8.36



Let N and T be midpoints of $\overline{HQ}$ and $\overline{AH}$, and call O the center of Γ. Let L be on the nine-point circle with $\angle HML = 90^\circ$. The negative inversion at H swapping Γ and nine-point circle maps A to F, K to L, and Q to M. As $\overline{LM} \parallel \overline{AQ}$ we just need to prove $LA = LQ$. But $\overline{MT}$ is a diameter, hence $LTNM$ is a rectangle, so $\overline{LT}$ passes through O (because the nine-point center is the midpoint of $\overline{OH}$).

Solution 8.37

Let P denote the center of ω_2. We are going to show that $\angle OFB = \angle OGB = 90^\circ$.

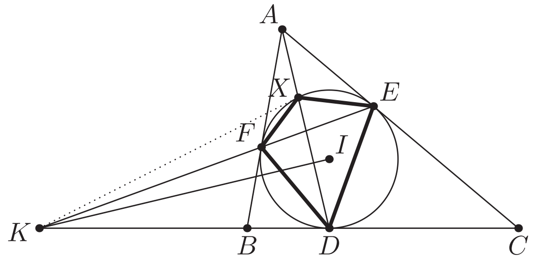


First, consider an inversion around ω_1 sending F to F^* and G to G^*. As this inversion fixes ω_2, we find that $\overline{AF^*}$ and $\overline{AG^*}$ are now the tangents to ω_2. Now it suffices to prove B lies on $\overline{F^*G^*}$, as it will then follow that $\angle OBF^* = \angle OBG^* = 90^\circ$.

Because ω_1 is orthogonal to ω_2, it follows B and A are inverses under a second inversion around ω_2. Since A is the intersection of the tangents at F^* and G^*, we also know the image of A under this second inversion is the midpoint of $\overline{F^*G^*}$. Thus it follows that B is the midpoint of $\overline{F^*G^*}$ as desired.

Solution 9.40

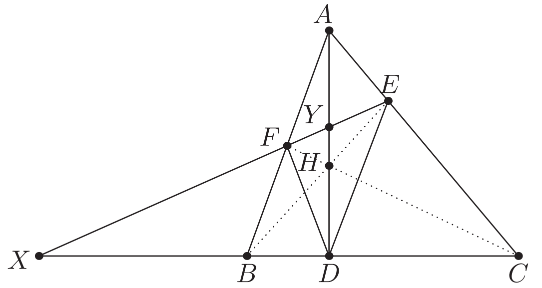
Let X denote the second intersection of $\overline{AD}$ with the incircle.



Since $\overline{AF}$ and $\overline{AE}$ are tangents to the incircle, we discover that $XFDE$ is a harmonic quadrilateral (by Lemma 9.9). Now K is the intersection of line EF and the tangent to D, so the fact that $XFDE$ is harmonic implies that $\overline{KX}$ is tangent to the incircle as well. Consequently $\overline{KI} \perp \overline{XD}$; in fact, K is the pole of line XD.

Solution 9.44

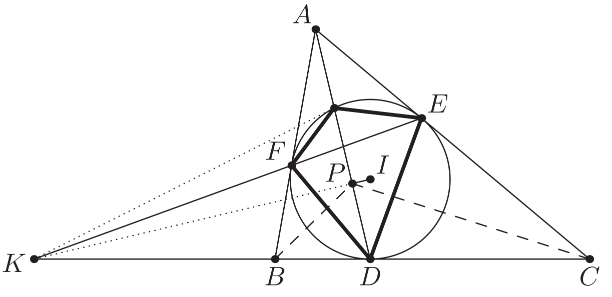
Let line EF meet BC again at X. Moreover, let line AH meet line EF at Y.



By Lemma 9.11 on $\triangle ABC$, we derive that $(X, D; B, C) = -1$; perspectivity at A gives $(X, Y; E, F) = -1$. (Alternatively, apply Lemma 9.11 on $\triangle AEF$.) In any case, since we know $\angle XDY = 90^\circ$, applying Lemma 9.18 shows that $\overline{DH}$ bisects $\angle FDE$.

Solution 9.46

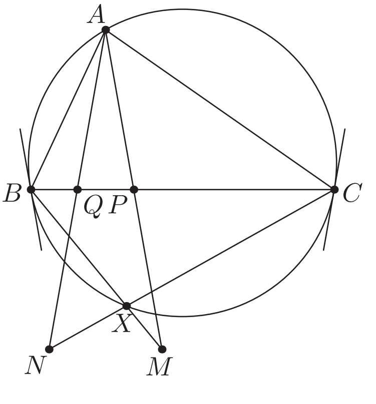
This is just an extension of Lemma 9.40. Again denote by K the intersection of ray IP with $\overline{BC}$.



In Lemma 9.40 we showed that $(K, D; B, C) = -1$ (this also follows from directly applying Lemma 9.11 to the cevians $\overline{AD}$, $\overline{BE}$, $\overline{CF}$, where E and F are the tangency points of the incircle to the opposite sides). Now observe that $\angle KPD = 90^\circ$, so Lemma 9.18 implies that $\overline{PD}$ bisects $\angle BPC$.

Solution 9.47

Let $\overline{BM}$ intersect the circumcircle again at X.



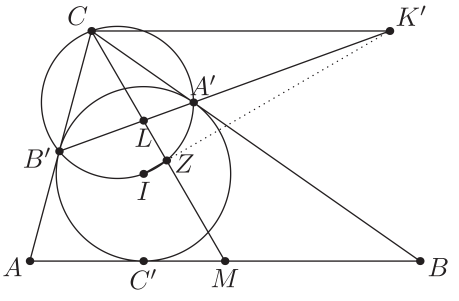
The angle conditions imply that the tangent to (ABC) at B is parallel to $\overline{AP}$. Let P_∞ be the point at infinity along line AP. Then

$$-1 = (A, M; P, P_\infty) \stackrel{B}{=} (A, X; B, C).$$

Similarly, if $\overline{CN}$ meets the circumcircle at Y then $(A, Y; B, C) = -1$ as well. Hence $X = Y$, which implies the problem condition.

Solution 9.49

Let M be the midpoint of $\overline{AB}$. Let Z be the foot of the perpendicular from I to $\overline{CM}$, and note that the points C, B', I, Z, A' all lie on a circle with diameter $\overline{CI}$. Let K' be on line $A'B'$ so that $\overline{K'C} \parallel \overline{AB}$. We prove that $\angle K'ZL$ is right, because this implies $K' = K$.



Notice that $(A, B; M, P_\infty)$ is harmonic, where P_∞ is the point at infinity along $\overline{AB}$. Taking perspectivity from C onto line $A'B'$ we observe that $(B', A'; L, K')$ is harmonic.

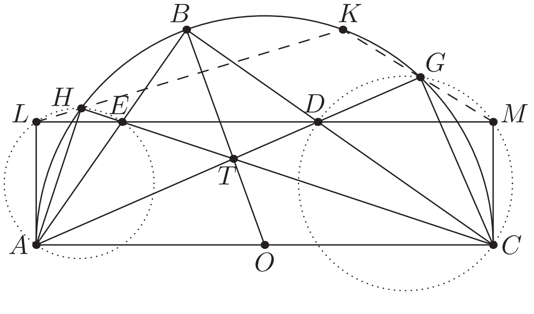
Now consider point Z. We know that $\measuredangle CZB' = \measuredangle CIB' = \measuredangle A'IC = \measuredangle A'ZC$, so $\overline{ZC}$ bisects $\angle A'ZB'$. Thus Lemma 9.18 applies and we conclude $\angle LZK' = 90^\circ$ as needed.

Solution 9.50

Refer to Figure 9.9A. Pascal's theorem on $AGEEBC$ shows that $\overline{BC} \cap \overline{GE}$ lies on d. Let G' be the reflection of G over $\overline{AB}$. Then applying Pascal's theorem to $CG'GEBB$ forces $\overline{CG} \cap \overline{BE}$ to lie on d, so the intersection must be the point F.

Solution 9.54

Set $T = \overline{AD} \cap \overline{CE}$, $O = \overline{BT} \cap \overline{AC}$, and $K = \overline{LH} \cap \overline{GM}$. We are going to ignore the condition that A, D, E, C is cyclic.



Now we can take a projective transformation that preserves the circumcircle of ABC and sends O to the center of the circle. In that case, $\overline{AC}$ is a diameter, and moreover T lies on the B-median of $\triangle ABC$, meaning that $\overline{DE} \parallel \overline{AC}$.

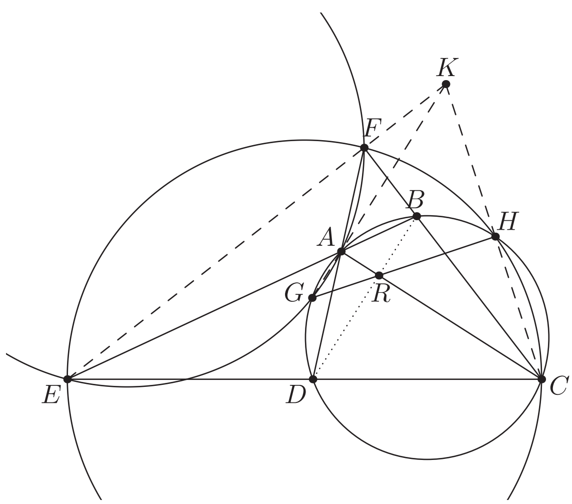
From this we deduce that $ALMC$ is a rectangle. Now we see that $ALHE$ and $DGMC$ are cyclic. From this we can use angle chasing to compute $\measuredangle HKG$ as

$$\begin{aligned}\measuredangle HKG = \measuredangle LKM &= -\measuredangle KML - \measuredangle MLK \\ &= -\measuredangle GMD - \measuredangle ELH \\ &= -\measuredangle GCD - \measuredangle EAH = -\measuredangle GCB - \measuredangle BAH \\ &= -\measuredangle GAB - \measuredangle BAH = -\measuredangle GAH = -\measuredangle GBH \\ &= \measuredangle HBG.\end{aligned}$$

Hence H, B, K, G are concyclic and we are done.

Solution 9.56

Let K be the radical center of ω, ω_1, ω_2, so that K is the intersection of $\overline{AG}$, $\overline{CH}$, and $\overline{EF}$. Let $R = \overline{AC} \cap \overline{GH}$. The problem is to prove that R lies on $\overline{BD}$. Hence by Brocard's theorem on $ABCD$, it suffices to check that the polar of R is line EF.

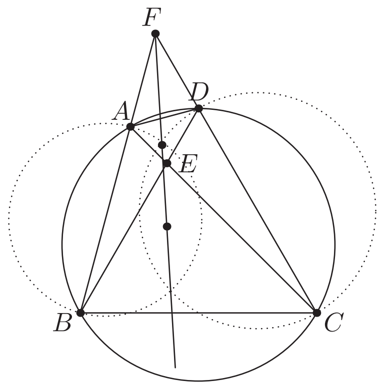


By applying Brocard's theorem on quadrilateral $ACGH$, we find that the polar of R is a line passing through the pole of $\overline{AC}$ and the point $K = \overline{AG} \cap \overline{CH}$. But the pole of $\overline{AC}$ lies on $\overline{EF}$ by Brocard's theorem on $ABCD$. Moreover, so does the point K by construction. Thus the pole of $\overline{AC}$ and the point K both lie on EF. Hence the polar of R really is $\overline{EF}$, and we are done.

Solution 10.19

Consider the circle ω_1 with diameter $\overline{AB}$ and the circle ω_2 with diameter $\overline{CD}$. Moreover, let ω be the circumcircle of $ABCD$.

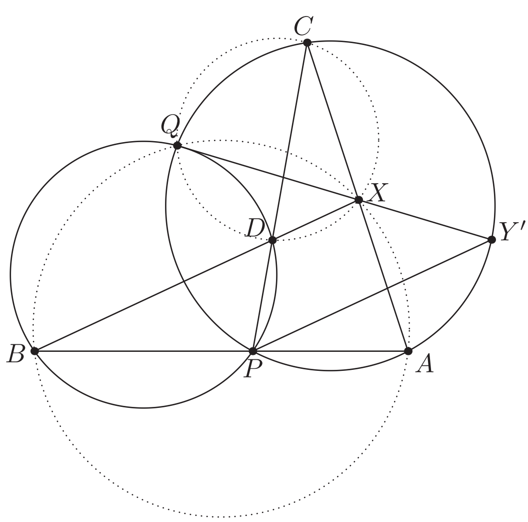
We saw already in the proof of Theorem 10.5 that the two orthocenters lie on the radical axis of ω_1 and ω_2 (i.e., the Steiner line of $ADBC$). Hence the problem is solved if we can



prove that F also lies on this radical axis. But this follows from the fact that F is actually the radical center of circles ω_1, ω_2 and ω.

Solution 10.20

Let Y' be the second intersection of ray QX with ω_1. We prove that $\overline{PY'} \parallel \overline{BD}$, which implies that Q, X, Y are collinear. (The point Z is handled similarly.)

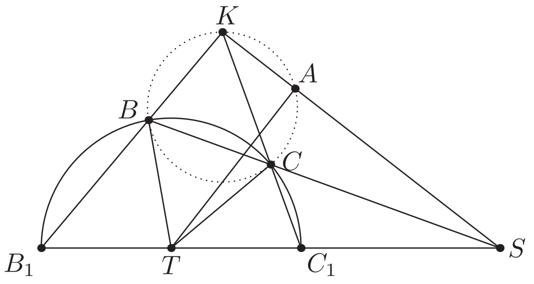


The given conditions imply that Q is the Miquel point of complete quadrilateral $DXAP$. Hence quadrilaterals $CQDX$ and $BQXA$ are cyclic. Therefore,

$$\measuredangle QY'P = \measuredangle QCP = \measuredangle QCD = \measuredangle QXD = \measuredangle QXB$$

which implies $\overline{PY'} \parallel \overline{BX}$.

Solution 10.22



Let K denote the intersection of $\overline{BB_1}$ and $\overline{CC_1}$. By angle chasing, we can check that

$$\angle BKC = \frac{1}{2}(180^\circ - \angle BTC) = \angle BAC.$$

So B, K, A, C are concyclic.

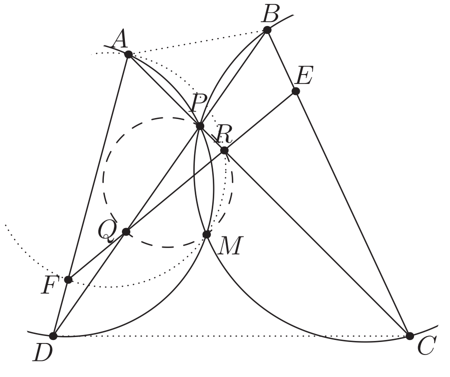
Consider Theorem 10.12 on quadrilateral B_1BCC_1. We know that

- A lies on (KBC)
- $\angle TAS = 90^\circ$
- $\angle BAC < 90^\circ$ since $\triangle ABC$ is given to be acute, so A lies outside of B_1BCC_1.

If we fix B_1BCC_1, it is easy to see that these conditions uniquely determine the point A. But the Miquel point of B_1BCC_1 also satisfies all three conditions. It follows that A must be the Miquel point, and it is now immediate that triangles ABC and AB_1C_1 are similar.

Solution 10.23

Let M be the Miquel point of complete quadrilateral $ADBC$; in other words, let M be the second intersection point of the circumcircles of $\triangle APD$ and $\triangle BPC$.



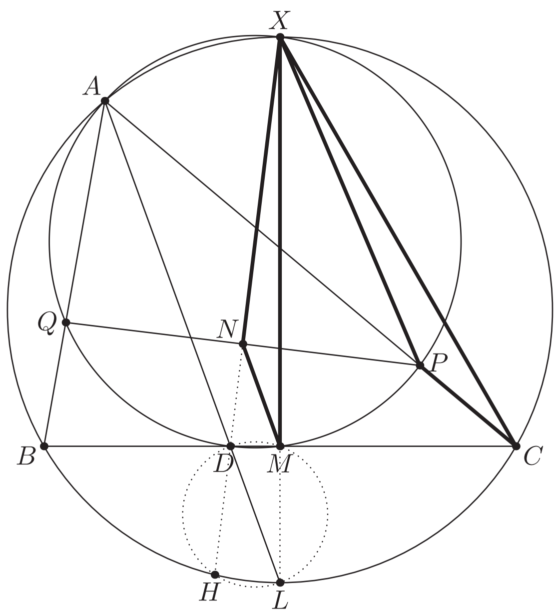
Since $\frac{AF}{AD} = \frac{CE}{CB}$, M is also the center of a spiral similarity which takes $\overline{FA}$ to $\overline{EC}$, thus it is the Miquel point of complete quadrilateral $FACE$. As $R = \overline{FE} \cap \overline{AC}$ we deduce $FARM$ is a cyclic quadrilateral.

Now look at complete quadrilateral $AFQP$. Since M lies on (DFQ) and (RAF), it follows that M is in fact the Miquel point of $AFQP$ as well. So M lies on (PQR).

Thus M is the fixed point that we wanted.

Solution 10.26

The main point of the problem is to prove that $\overline{MN} \parallel \overline{AD}$. First, denote by X the point diametrically opposite L on (ABC).



Since $\measuredangle XAD = \measuredangle XMD = 90^\circ$, it follows that A, M, D, X are concyclic. Thus X is the Miquel point of complete quadrilateral $PQBC$, and the center of the spiral similarity taking $\overline{QP}$ to $\overline{BC}$. Thus it is also the center of the spiral similarity taking $\overline{NP}$ to $\overline{MC}$. Equivalently, X is the center of the spiral similarity taking $\overline{NM}$ to $\overline{PC}$.

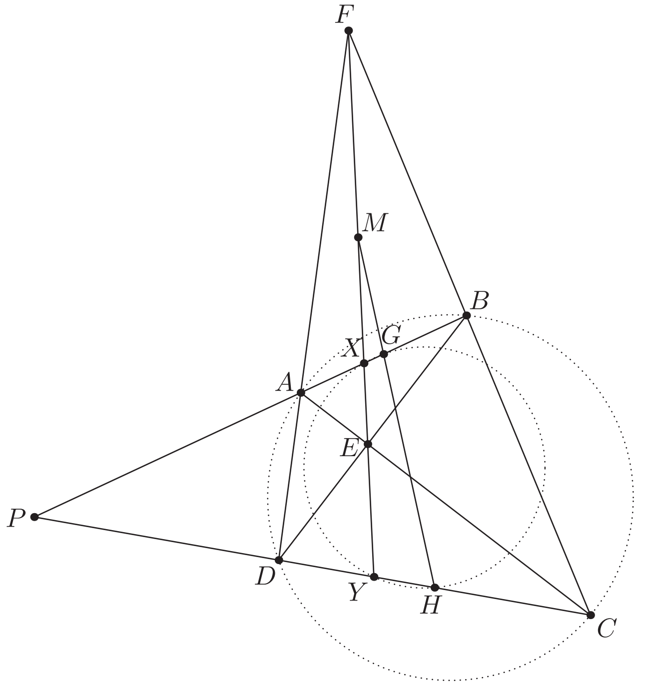
That implies $\triangle XNM$ and $\triangle XPC$ are similar with the same orientation, whence

$$\measuredangle NMX = \measuredangle PCX = \measuredangle ACX = \measuredangle ALX$$

implying that $\overline{MN} \parallel \overline{AL}$. Thus, $\measuredangle HMN = \measuredangle HDL = \measuredangle HML$ and we win.

Solution 10.29

Let M be the midpoint of $\overline{EF}$. Then M, G, H lie on the Gauss line of complete quadrilateral $ADBC$. Let $P = \overline{AB} \cap \overline{CD}$ and let line EF meet $\overline{AB}$ and $\overline{CD}$ at X and Y, respectively.



We have harmonic bundles

$$(X, Y; E, F) = (P, X; A, B) = (P, Y; D, C) = -1.$$

Using Lemma 9.17, we find

$$PX \cdot PG = PA \cdot PB = PD \cdot PC = PY \cdot PH.$$

Hence X, Y, G, H are concyclic.

Now, using Lemma 9.17 again on $(P, E; X, Y) = -1$ gives

$$ME^2 = MX \cdot MY = MG \cdot MH$$

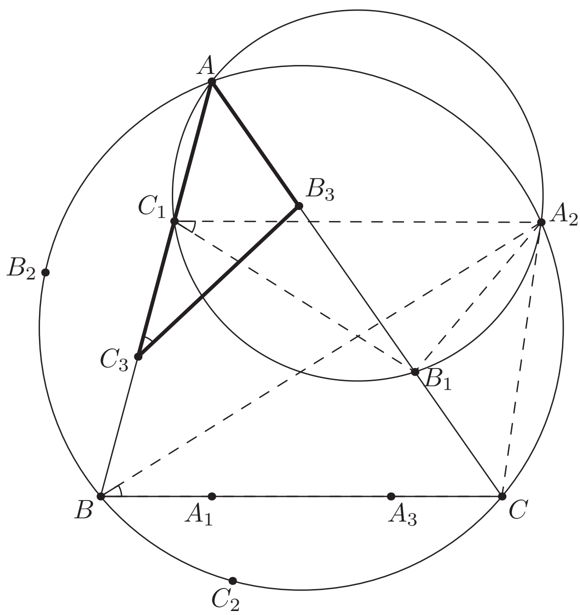
which gives the desired conclusion.

Solution 10.30

We are going to prove that

$$\measuredangle AC_3B_3 = \measuredangle A_2BC.$$

This solves the problem, because the analogous calculation $\measuredangle BC_3A_3 = \measuredangle B_2AC$ implies $\measuredangle A_3C_3B_3 = \measuredangle A_3C_3A + \measuredangle AC_3B_3 = \measuredangle A_3C_3B + \measuredangle AC_3B_3$, which gives $\measuredangle CAB_2 + \measuredangle A_2BC = \measuredangle A_2C_2C + \measuredangle CC_2B_2 = \measuredangle A_2C_2B_2$.



By spiral similarity at A_2, we deduce that $\triangle A_2C_1B \sim \triangle A_2B_1C$. Hence

$$\frac{A_2B}{A_2C} = \frac{A_2C_1}{A_2B_1} = \frac{C_1B}{B_1C} = \frac{AC_3}{AB_3}.$$

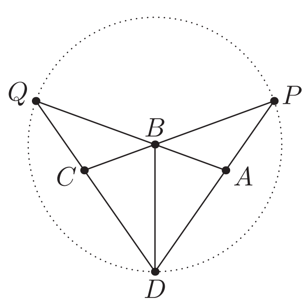
Moreover, $\measuredangle BA_2C = \measuredangle BAC = \measuredangle C_3AB_3$. We can check that A_2 lies on the same side of A as $\overline{BC}$ since B_1 and C_1 are constrained to lie on the sides of the triangle. So we can deduce $\angle C_3AB_3 = \angle BA_2C$. That implies $\triangle A_2BC \sim \triangle AC_3B_3$. Thus $\measuredangle AC_3B_3 = \measuredangle A_2BC$, completing the proof.

C.4 Solutions to Chapter 11

Solution 11.0

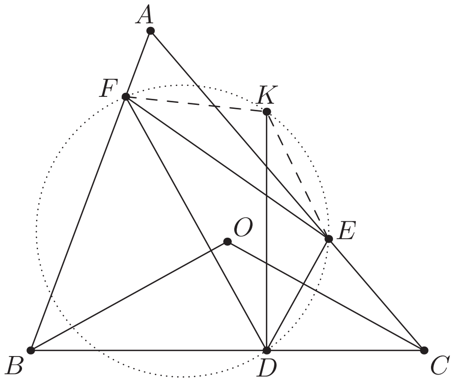
Have fun!

Solution 11.1



Let $P = \overline{AD} \cap \overline{BC}$, $Q = \overline{AB} \cap \overline{CD}$. Now $2\angle ADB = \angle CBD = \angle BPD + \angle PDB$, meaning $\angle BPD = \angle BDP$ and $BP = BD$. Similarly, $BQ = BD$. Now $BP = BQ$ and $BC = BA$ give $\triangle QBC \cong \triangle PBA$; from here the solution follows readily.

Solution 11.2

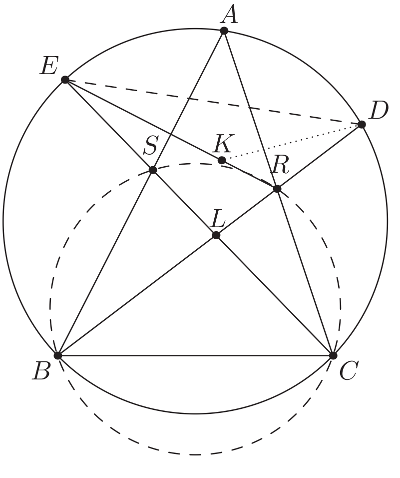


First, note $\measuredangle EDF = 180^\circ - \measuredangle BOC = 180^\circ - 2A$, so $\measuredangle FDE = 2A$. Observe that $\measuredangle FKE = 2A$ as well; hence $KFDE$ is cyclic. Hence

$$\begin{aligned}\measuredangle KDB &= \measuredangle KDF + \measuredangle FDB \\ &= \measuredangle KEF + (90^\circ - \measuredangle DBO) \\ &= (90^\circ - A) + (90^\circ - (90^\circ - A)) \\ &= 90^\circ.\end{aligned}$$

and the proof ends here.

Solution 11.3



Solution 1. Angle chasing reveals $\angle DCA = \angle ACE = \angle DBA = \angle ABE$.

First, we claim that $BE = BR = BC$. Indeed, construct a circle with radius $BE = BR$ centered at B, and notice that $\angle ECR = \frac{1}{2}\angle EBR$, implying that it lies on the circle.

Now, CA bisects $\angle ECD$ and DB bisects $\angle EDC$, so R is the incenter of $\triangle CDE$. Then, K is the incenter of $\triangle LED$, so

$$\angle ELK = \frac{1}{2}\angle ELD = \frac{1}{2}\left(\frac{\widehat{ED} + \widehat{BC}}{2}\right) = \frac{1}{2}\frac{\widehat{BED}}{2} = \frac{1}{2}\angle BCD.$$

Solution 2. Because

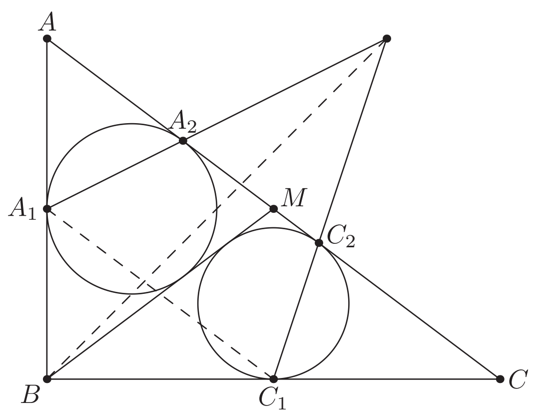
$$\measuredangle EBA = \measuredangle ECA = \measuredangle SCR = \measuredangle SBR = \measuredangle ABR,$$

$\overline{BA}$ bisects $\angle EBR$. Then by symmetry $\angle BEA = \angle BRA$, so

$$\measuredangle BCR = \measuredangle BCA = \measuredangle BEA = -\measuredangle BRA = -\measuredangle BRC$$

and hence it follows that $BE = BR = BC$. Now we proceed as in the first solution.

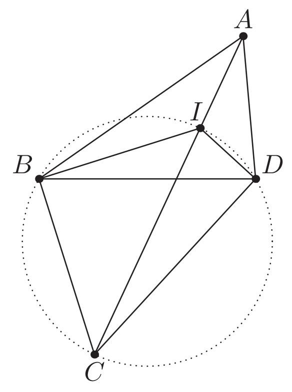
Solution 11.4



Because $MA = MB = MC$, A_1 and C_1 are merely the midpoints of $\overline{AB}$ and $\overline{BC}$; in particular, $\overline{A_1C_1} \parallel \overline{AC}$. Moreover, $\angle AA_1A_2 = \angle AA_2A_1 = \angle C_1A_1A_2$ and so $\overline{A_1A_2}$ is the external angle bisector of $\angle A_1$ in triangle A_1BC_1. Similarly, $\overline{C_1C_2}$ is the external angle bisector of $\angle C_1$. Hence they intersect at the excenter, which lies on the B-bisector of this triangle.

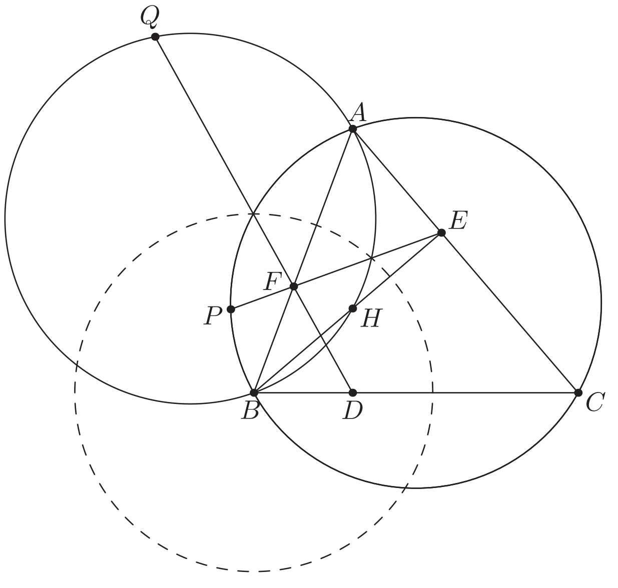
Solution 11.5

The following diagram is not drawn to scale.



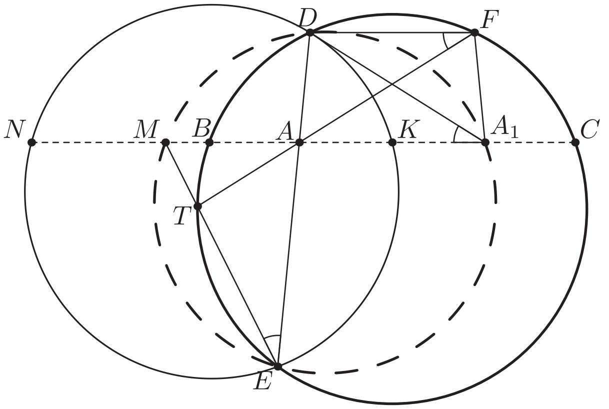
Let I denote the incenter of $\triangle ABD$. Then quadrilateral $IBCD$ is cyclic since $\angle DIB = 90^\circ + \frac{1}{2}\angle DAB = 145^\circ$. Hence we obtain $\angle IBD = \angle ICD = 180^\circ - (55^\circ + 105^\circ) = 20^\circ$ and $\angle ABD = 40^\circ$.

Solution 11.6



Of course H lies on γ (for example, by Lemma 1.17). Now consider an inversion at B with power $\sqrt{BH \cdot BE} = \sqrt{BF \cdot BA} = \sqrt{BD \cdot BC}$. It swaps the three pairs F and A, D and C, and H and E. That means it swaps the circle γ with the line EF and the circle ω with line DF. It follows that P and Q map to each other and we are done.

Solution 11.7



Let K be the midpoint of $\overline{BC}$ and let A_1 be the reflection of A over K. Because F is the reflection of D over the perpendicular bisector of $\overline{BC}$, we find that DFA_1A is an isosceles trapezoid. Then,

$$\measuredangle MED = \measuredangle TED = \measuredangle TFD = \measuredangle AFD = \measuredangle AA_1D = \measuredangle MA_1D.$$

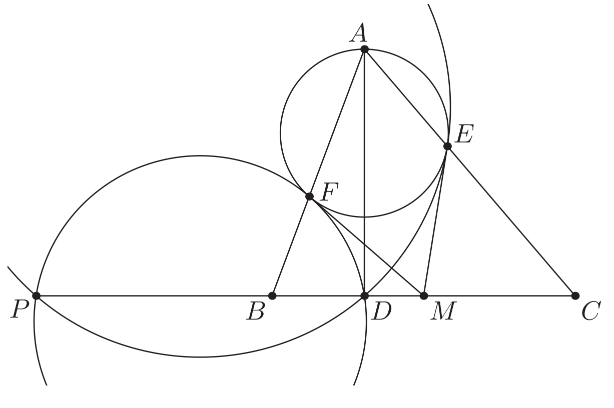
Therefore, MDA_1E is cyclic. Now, by power of a point, we see that

$$AD \cdot AE = AM \cdot AA_1 = 2AM \cdot AK = AN \cdot AK.$$

Therefore, $DKEN$ is cyclic, as desired.

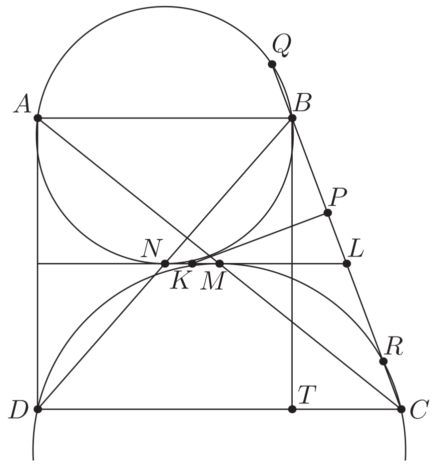
Solution 11.8

Let M denote the midpoint of $\overline{BC}$.



By Lemma 1.44, $\overline{ME}$ and $\overline{MF}$ are tangents to ω (and hence to ω_1, ω_2), so M is the radical center of ω, ω_1, ω_2. Now consider the radical axis of ω_1 and ω_2. It passes through D and M, so it is line BC, and we are done.

Solution 11.9



Let $AB = 2x$, $CD = 2y$, and assume without loss of generality that $x < y$. Let L be the midpoint of $\overline{BC}$ and denote $BC = 2\ell$. Let P be the midpoint of $\overline{QR}$. Let T be the foot of B on $\overline{DC}$.

Since N is the midpoint of the hypotenuse of $\triangle ABD$, it follows that $AN = BN$. Since $\overline{MN} \parallel \overline{AB}$, we see that $\overline{MN}$ is tangent to (ABN). Similarly, it is tangent to (BCM).

Noting that $LM = \frac{1}{2}AB$ via $\triangle ABC$, we obtain

$$LR \cdot LC = LM^2 = \left(\frac{1}{2}AB\right)^2 = x^2 \Rightarrow LR = \frac{x^2}{\ell}.$$

Similarly, $LQ = \frac{y^2}{\ell}$. Then,

$$PL = \frac{LQ - LR}{2} = \frac{y^2 - x^2}{2\ell} \text{ and } KL = \frac{ML + NL}{2} = x + y.$$

But then we find that

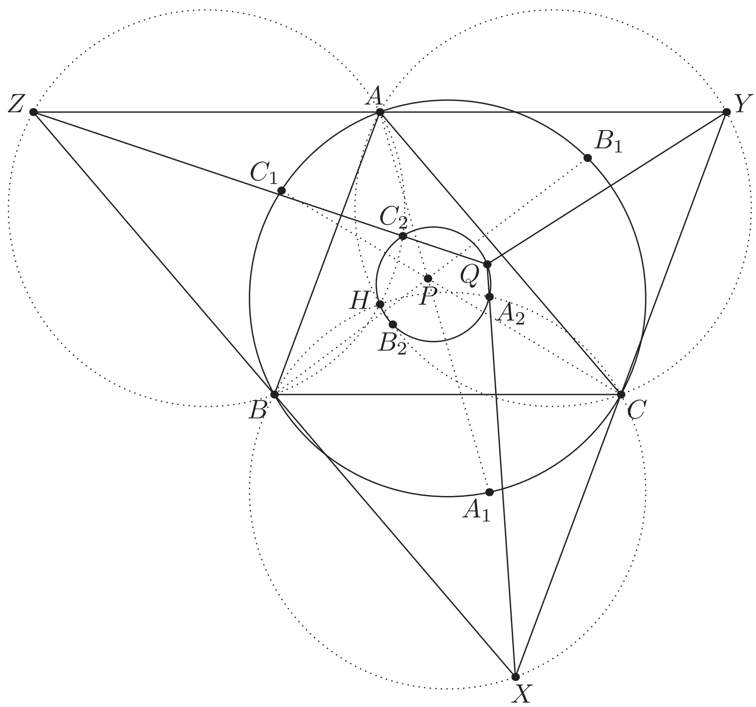
$$\frac{KL}{PL} = \frac{\frac{y^2 - x^2}{2\ell}}{x + y} = \frac{y - x}{2\ell} = \frac{TC}{BC}.$$

Combined with $\angle KLP = \angle BCT$, we find that $\triangle KLP \sim \triangle BCT$. Therefore, $\angle KPL = \angle BTC = 90^\circ$. But P is the midpoint of $\overline{QR}$, so $KQ = KR$.

Solution 11.10

Construct parallelograms $XCAB$, $YABC$, and $ZBCA$. By Ceva's theorem in trigonometric form on triangle ABC and point P, we know that

$$\frac{\sin \angle BAP}{\sin \angle PAC} \frac{\sin \angle CBP}{\sin \angle PBA} \frac{\sin \angle ACP}{\sin \angle PCB} = 1.$$



But $\angle PAC = \angle A_1AC = \angle CXA_2$, since minor arcs A_1C and A_2C are identical. So the above rewrites as

$$\frac{\sin \angle BXA_2}{\sin \angle CXA_2} \frac{\sin \angle CYB_2}{\sin \angle AYB_2} \frac{\sin \angle AZC_2}{\sin \angle BZC_2} = 1.$$

So rays XA_2, YB_2, ZC_2 concur at some point, say Q.

Let H be the orthocenter of triangle ABC. We claim that H is the fixed point, and that in fact, the three points lie on a circle with diameter $\overline{HQ}$. Indeed, note that A_2 lies on the reflection of (ABC) over $\overline{BC}$, which is a circle with diameter $\overline{HX}$, whence

$$\measuredangle HA_2X = \measuredangle HA_2Q = 90^\circ$$

as desired.

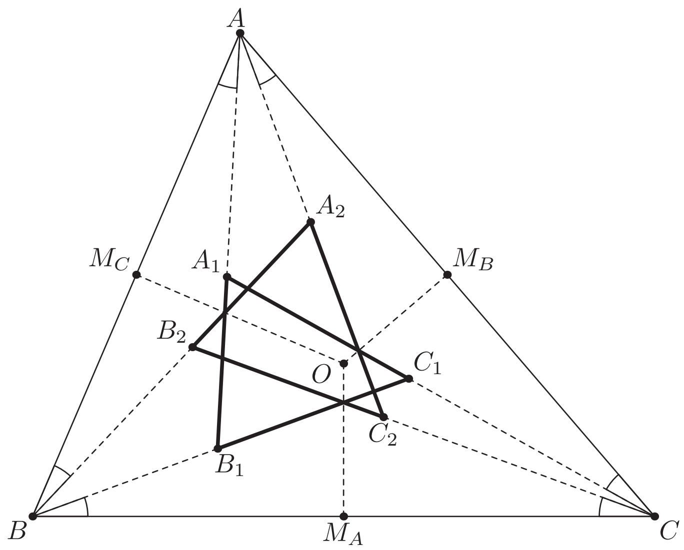
Solution 11.11

Easy angle chasing gives

$$\angle B_2A_2C_2 = \angle ABA_2 + \angle BAA_2 = \angle BAC.$$

Similar calculations yield that $\triangle A_1B_1C_1 \sim \triangle A_2B_2C_2 \sim \triangle ABC$.

Now, let O be the circumcenter of $\triangle ABC$. Then O lies on the angle bisector of the angle formed by lines B_2C_2 and B_1C_1; namely, the line through O perpendicular to $\overline{BC}$. (Note that $\angle B_1BC = \angle C_2CB$, giving an isosceles triangle.) Let d_a denote the common distance from O to lines B_2C_2 and B_1C_1. Define d_b and d_c analogously.



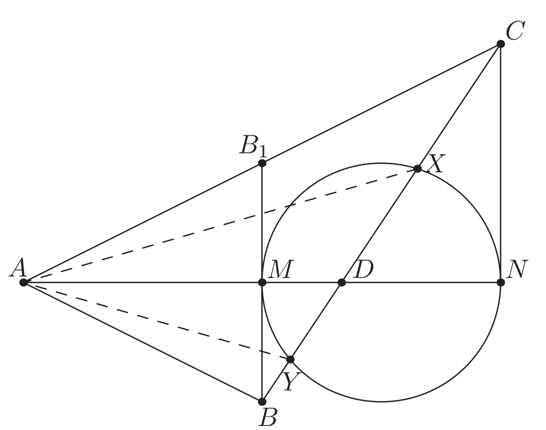
Then, since $\triangle A_1B_1C_1$ is similar to $\triangle A_2B_2C_2$, we observe that O must have the same barycentric coordinates with respect to $\triangle A_1B_1C_1$ and $\triangle A_2B_2C_2$, namely

$$\begin{aligned}&(d_a \cdot B_1C_1 : d_b \cdot C_1A_1 : d_c \cdot A_1B_1)\\ =&(d_a \cdot B_2C_2 : d_b \cdot C_2A_2 : d_c \cdot A_2B_2)\,.\end{aligned}$$

So O corresponds to the same point in both triangles. The congruence of the pedal triangles is then enough to deduce that $\triangle A_1B_1C_1$ is congruent to $\triangle A_2B_2C_2$.

Solution 11.12

Assume without loss of generality that $AB < AC$.



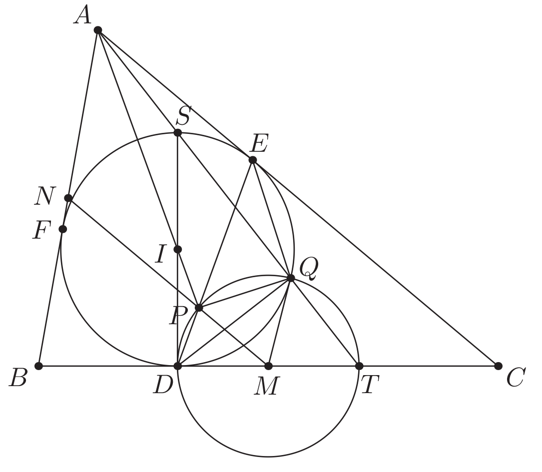
Let B_1 be the reflection of B over M (which is on $\overline{AC}$) and let P_∞ be the point at infinity along $\overline{BM} \parallel \overline{CN}$. Evidently

$$-1 = (B_1, B; M, P_\infty) \stackrel{C}{=} (A, D; M, N).$$

But $\angle MYN = \angle MXN = 90^\circ$, so by Lemma 9.18, we find that M is the incenter of $\triangle AXY$; hence $\angle XAM = \angle YAM$, and hence $\angle BAX = \angle CAY$ as desired.

Solution 11.13

Assume without loss of generality that $AB < AC$. We show that in this case, $\angle PQE = 90^\circ$.



First, we claim that D, P, E are collinear. Let N be the midpoint of $\overline{AB}$. Let P' be the intersection of the $\overline{MN}$, $\overline{DE}$, and ray AI, as in Lemma 1.45. Then P' lies inside $\triangle ABC$ and moreover $\triangle DP'M \sim \triangle DEC$, so $MP' = MD$. This is enough to imply that $P' = P$, proving the claim.

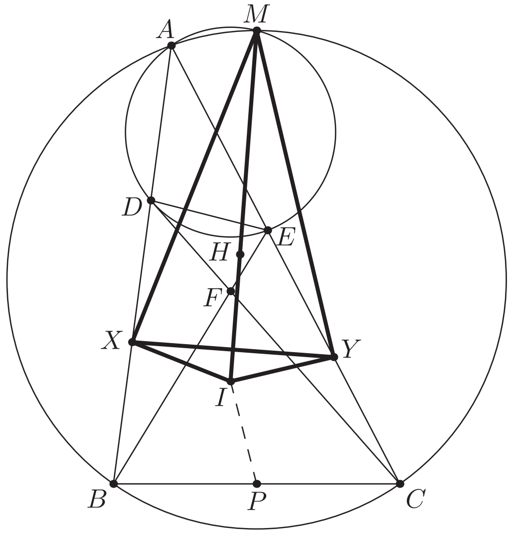
Let S be the point diametrically opposite D on the incircle, which is also the second intersection of $\overline{AQ}$ with the incircle. Let $T = \overline{AQ} \cap \overline{BC}$. Then T is the contact point of the A-excircle (Lemma 4.9); consequently, $MD = MP = MT$, and we obtain a circle with diameter $\overline{DT}$. Since $\measuredangle DQT = \measuredangle DQS = 90^\circ$ we have Q on this circle as well.

As $\overline{SD}$ is tangent to the circle with diameter $\overline{DT}$, we obtain $\measuredangle PQD = \measuredangle PDS = \measuredangle EDS = \measuredangle EQS$. Since $\measuredangle DQS = 90^\circ$, $\measuredangle PQE = 90^\circ$ too.

Solution 11.14

Evidently D and E are the reflections of C and B over $\overline{BI}$ and $\overline{CI}$, respectively. Denote by X and Y the midpoints of $\overline{BD}$ and $\overline{CE}$, and let P be the midpoint of $\overline{BC}$. Because of the reflections, we have that $IX = IP = IY$.

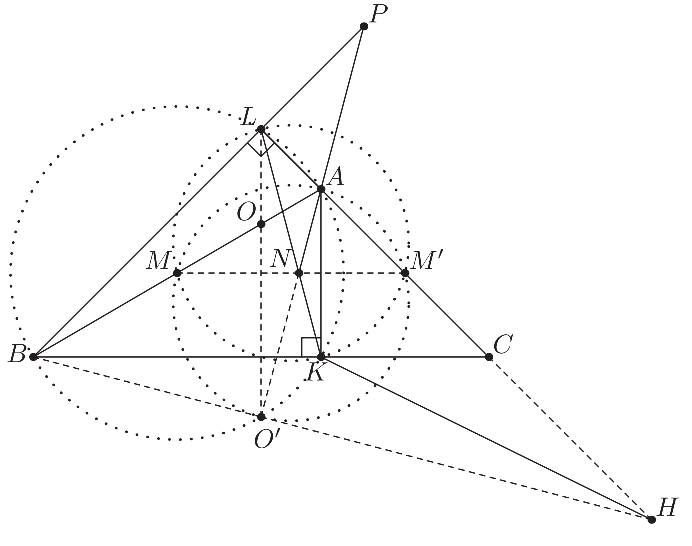
Next, consider the second intersection T of (ABC) and (ADE). It is the center of the spiral similarity that maps $\overline{BD}$ to $\overline{CE}$. But then the map must actually be a congruence as $BD = CE$, so $TB = TC$. Since T is on (ABC), and because we require $\triangle TBD$ and



$\triangle TCE$ to be similarly oriented, this implies $T = M$. Hence $MX = MY$; therefore $\overline{MI}$ is the perpendicular bisector of $\overline{XY}$.

Now $\overline{XY}$ is the Gauss line of complete quadrilateral $BEDC$. Since I is the orthocenter of triangle FBC, line $\overline{MI}$ is the Steiner line (since the Steiner and Gauss lines are perpendicular), which by definition passes through H.

Solution 11.15



Let M' be the midpoint of $\overline{AC}$ and let O' be the circumcenter of $\triangle ABC$. Then $KMLM'$ is cyclic (nine-point circle), as is $AMO'M'$ (since $\angle MOA = \angle MM'A = 45°$).

Also, $\angle BO'A = 90^\circ$, so O' lies on the circle with diameter $\overline{AB}$. Then N is the radical center of these three circles; hence A, N, O' are collinear.

Now applying Brocard's theorem to quadrilateral $BLAO'$, we find that M is the orthocenter of the OPH', where $H' = \overline{LA} \cap \overline{BO'}$. Hence H' is the orthocenter of $\triangle MOP$, whence $H = H' = \overline{AC} \cap \overline{BO'}$.

Now we know that

$$\frac{AH}{HC} = \frac{c^2(a^2+b^2-c^2)}{a^2(b^2+c^2-a^2)}$$

where the ratio is directed as in Menelaus's theorem. Cancelling a factor of 280^2 we can compute:

$$\frac{AH}{HC} = \frac{c^2(a^2+b^2-c^2)}{a^2(b^2+c^2-a^2)} = \frac{338(576+98-338)}{576(98+338-576)} = -\frac{169}{120}.$$

Therefore,

$$\begin{aligned} \frac{AC}{HC} &= 1 + \frac{AH}{HC} = -\frac{49}{120} \\ \Rightarrow |HC| &= \frac{120}{49} \cdot 1960\sqrt{2} = 4800\sqrt{2}. \end{aligned}$$

Now applying the law of cosines to $\triangle KCH$ with $\angle KCH = 135^\circ$ yields

$$\begin{aligned} HK^2 &= KC^2 + CH^2 - 2KC \cdot CH \cdot \cos 135^\circ \\ &= 1960^2 + \left(4800\sqrt{2}\right)^2 - 2(1960)\left(4800\sqrt{2}\right)\left(-\frac{1}{\sqrt{2}}\right) \\ &= 40^2\left(49^2 + 2\cdot 120^2 + 2\cdot 49\cdot 120\right) \\ &= 1600 \cdot 42961 \\ &= 68737600. \end{aligned}$$

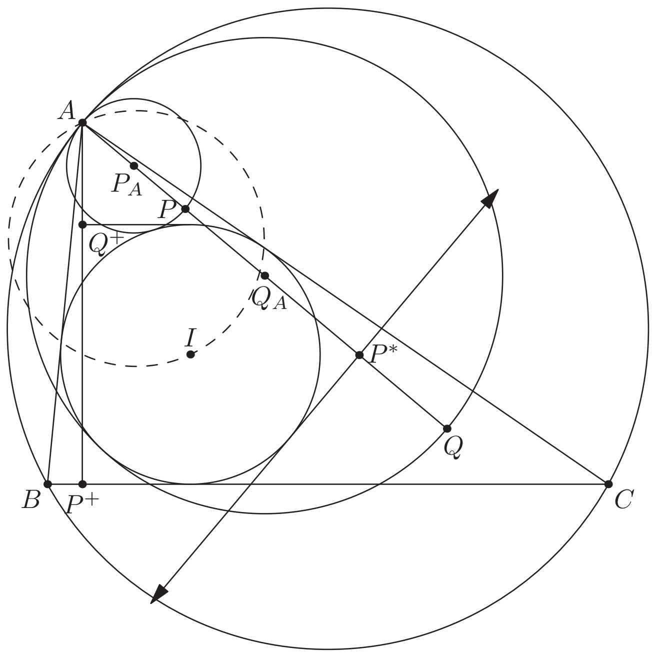
Solution 11.16

It turns out we can compute $P_A Q_A$ explicitly. Let us invert around A with radius $s-a$ (hence fixing the incircle) and then compose this with a reflection around the angle bisector of $\angle BAC$. We let this operation send a point X to X^* then to X^+. We overlay this inversion with the original diagram.

Let $P_A Q_A$ meet ω_A again at P and S_A again at Q. Now observe that ω_A^* is a line parallel to S^*; that is, it is perpendicular to $\overline{PQ}$. Moreover, it is tangent to $\omega^* = \omega$.

Now upon the reflection, we find that $\omega^+ = \omega^* = \omega$, but line $\overline{PQ}$ gets mapped to the altitude from A to $\overline{BC}$, since $\overline{PQ}$ originally contained the circumcenter O (isogonal to the orthocenter). But this means that ω_A^* is none other than the $\overline{BC}$! Hence P^+ is actually the foot of the altitude from A onto $\overline{BC}$.

By similar work, we find that Q^+ is the point on $\overline{AP^+}$ such that $P^+Q^+ = 2r$.



Now we can compute all the lengths directly. We have that

$$AP_A = \frac{1}{2}AP = \frac{(s-a)^2}{2AP^+} = \frac{1}{2}(s-a)^2 \cdot \frac{1}{h_a}$$

and

$$AQ_A = \frac{1}{2}AQ = \frac{(s-a)^2}{2AQ^+} = \frac{1}{2}(s-a)^2 \cdot \frac{1}{h_a - 2r}$$

where $h_a = \frac{2K}{a}$ is the length of the A-altitude, with K the area of ABC as usual. Now it follows that

$$P_A Q_A = \frac{1}{2}(s-a)^2 \left(\frac{2r}{h_a(h_a - 2r)} \right).$$

This can be simplified, as

$$h_a - 2r = \frac{2K}{a} - \frac{2K}{s} = 2K \cdot \frac{s-a}{as}.$$

Hence

$$P_A Q_A = \frac{a^2 rs(s-a)}{4K^2} = \frac{a^2(s-a)}{4K}.$$

Hence, the problem is just asking us to show that

$$a^2b^2c^2(s-a)(s-b)(s-c) \le 8(RK)^3.$$

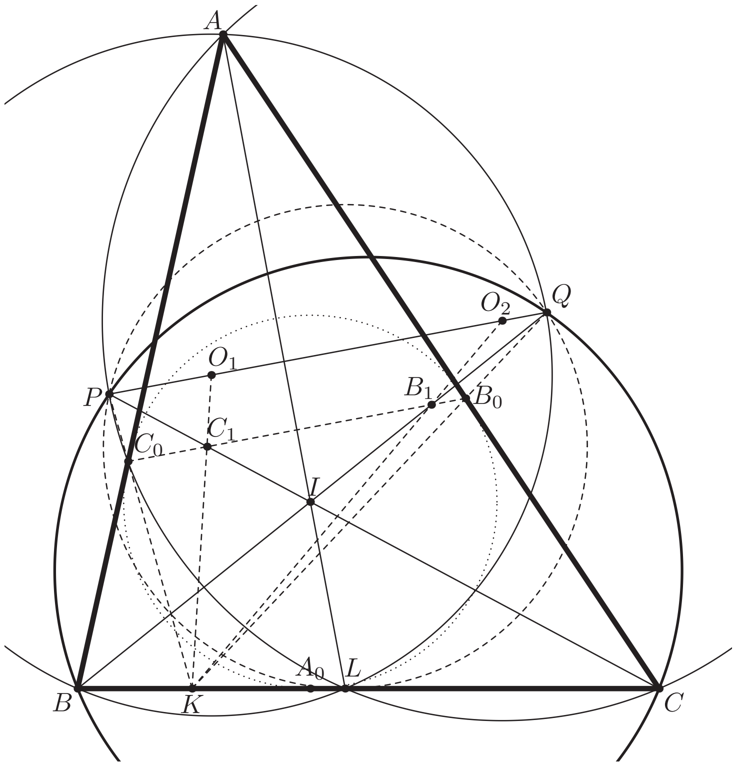
Using $abc = 4RK$ and $(s-a)(s-b)(s-c) = \frac{1}{s}K^2 = rK$, we find that this becomes

$$2(s-a)(s-b)(s-c) \le RK \Leftrightarrow 2r \le R$$

which follows immediately from Lemma 2.22. Alternatively, one may rewrite this as Schur's Inequality in the form

$$abc \geq (-a+b+c)(a-b+c)(a+b-c).$$

Solution 11.17



Let the incircle touch $\overline{BC}$ at A_0. First, note that B_1 and C_1 lie on $\overline{B_0C_0}$ by Lemma 1.45. Next, Q lies on (ABL), since $\overline{BI}$ is an internal angle bisector and we know that $QA = QL$ (this is Lemma 1.18). Similarly, P lies on (ACL).

We claim that $\triangle A_0B_0C_0$ and $\triangle LQP$ are homothetic (where A_0 is the tangency point of the incircle on $\overline{BC}$). Since $\overline{B_0C_0}$ and $\overline{PQ}$ are both perpendicular to $\overline{AL}$, we have $\overline{B_0C_0} \parallel \overline{PQ}$. Also, $\angle C_0A_0B = \frac{180^\circ - B}{2}$, and

$$\angle PLB = \angle PAC = \angle PAL + \angle LAC = \frac{1}{2}C + \frac{1}{2}A = \frac{180^\circ - B}{2}$$

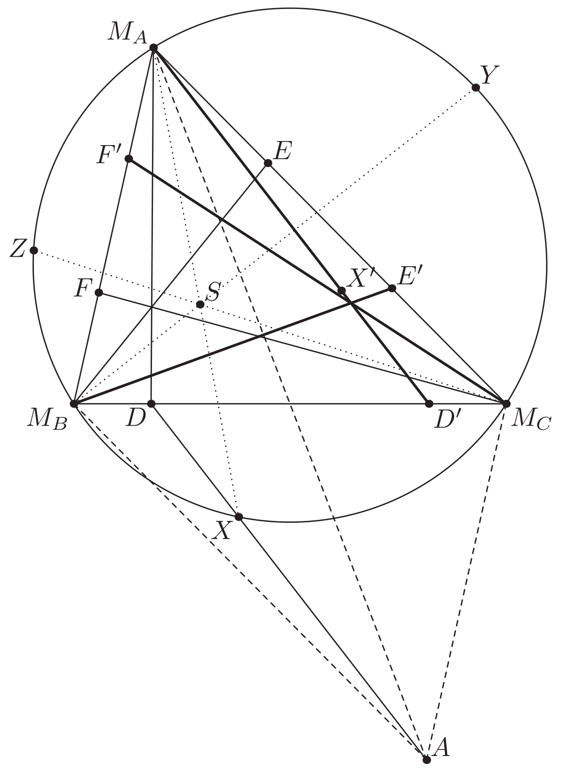
which shows that $\overline{C_0A_0} \parallel \overline{PL}$. Similarly, $\overline{B_0A_0} \parallel \overline{LQ}$.

Hence $\triangle A_0B_0C_0$ and $\triangle LQP$ are homothetic. Let K be the center of homothety; because $K \in \overline{LA_0} = \overline{BC}$, $\overline{QB_0}$ and $\overline{BC}$ are concurrent.

It remains to show $\overline{KC_1}$ passes through O_1. Let O_1' be the intersection of $\overline{PQ}$ and $\overline{C_1K}$. Then O_1' is the image of C_1. Since $B_0C_1 = A_0C_1$, it follows that $QO_1' = LO_1'$. But $\overline{PQ}$ happens to be the perpendicular bisector of $\overline{AL}$, so in fact $O_1'A = O_1'Q = O_1'L$. Hence

O_1' is the circumcenter of (ABL); that is, $O_1 = O_1'$. Similarly $O_2 = O_2'$ and the proof is complete.

Solution 11.18



Let AX meet $\overline{M_B M_C}$ at D and let X reflected over the midpoint of $\overline{M_B M_C}$ be X'. Let Y', Z', E, F be similarly defined.

By cevian nest (Theorem 3.23) it suffices to prove that $\overline{M_A D}$, $\overline{M_B E}$, $\overline{M_C F}$ are concurrent. Taking the isotomic conjugate and recalling that $M_A M_B A M_C$ is a parallelogram, we see that it suffices to prove $\overline{M_A X'}$, $\overline{M_B Y'}$, $\overline{M_C Z'}$ are concurrent.

We now use barycentric coordinates on $\triangle M_A M_B M_C$. Let

$$S = \left(a^2 S_A + t : b^2 S_B + t : c^2 S_C + t\right)$$

(possibly $t = \infty$ if S is the centroid). Let $v = b^2 S_B + t$, $w = c^2 S_C + t$. Hence

$$X = \left(-a^2 vw : (b^2 w + c^2 v)v : (b^2 w + c^2 v)w\right).$$

Consequently,

$$X' = \left(a^2 vw : -a^2 vw + (b^2 w + c^2 v)w : -a^2 vw + (b^2 w + c^2 v)v\right).$$

We can compute

$$b^2 w + c^2 v = (bc)^2(S_B + S_C) + (b^2 + c^2)t = (abc)^2 + 2t$$

so

$$-a^2v + b^2w + c^2v = (b^2 + c^2) + (abc)^2 - (ab)^2S_B - a^2t = S_A(ab + t).$$

Thus

$$X' = \left(a^2vw : S_A(b^2S_B + t)(ab + t) : S_A(c^2S_C + t)(ac + t)\right).$$

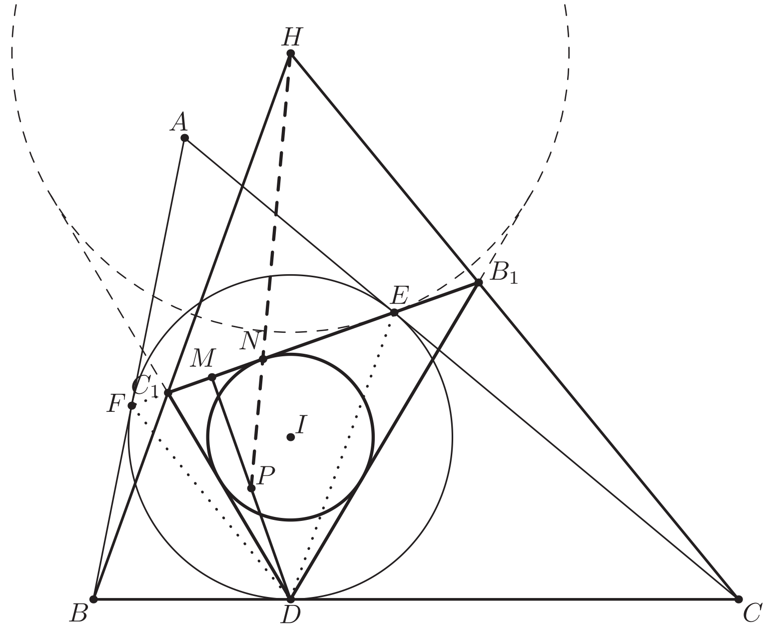
Similarly,

$$Y' = \left(S_B(a^2S_A + t)(ba + t) : b^2wu : S_B(c^2S_C + t)(bc + t)\right)$$
$$Z' = \left(S_C(a^2S_A + t)(ca + t) : S_C(b^2S_B + t)(cb + t) : c^2uv\right).$$

Now we are done by Ceva's theorem.

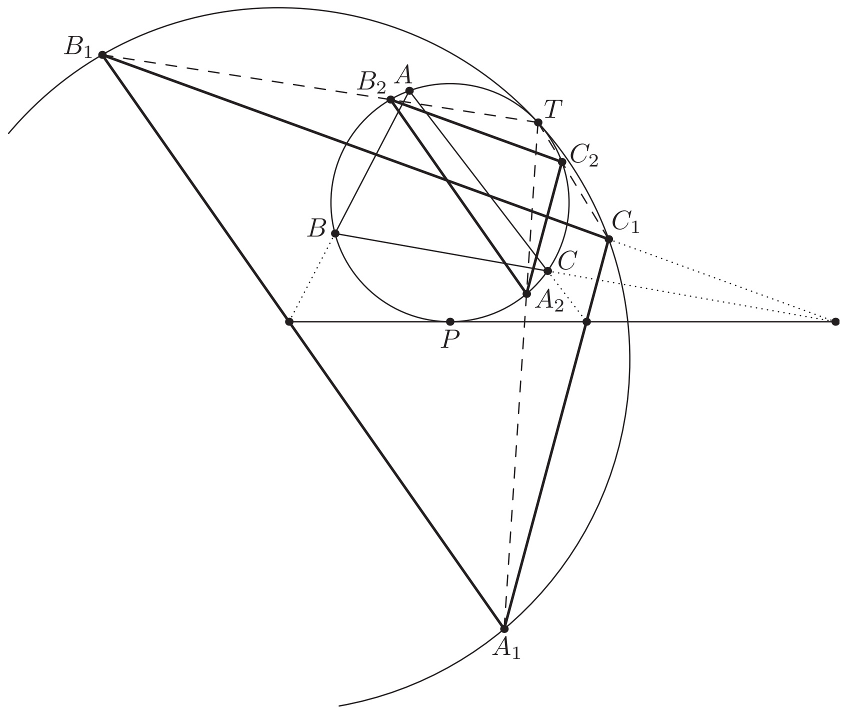
Solution 11.19

Let N be the midpoint of $\overline{EF}$, and set $B_1 = \overline{EF} \cap \overline{HC}$, $C_1 = \overline{EF} \cap \overline{HB}$. Focus on triangle DB_1C_1.



By Lemma 1.45, $\triangle DBC_1$ is the orthic triangle of $\triangle HBC$. Moreover, N is the tangency point of its incircle with $\overline{B_1C_1}$. In addition, H is the D-excenter (via Lemma 4.6). Then Lemma 4.14 implies P, N, and H are collinear.

Solution 11.20



This is a hard problem with many beautiful solutions. The following solution is not very beautiful but not too hard to find during an olympiad, as the only major insight it requires is the construction of A_2, B_2, and C_2.

We apply complex numbers with ω the unit circle and $p = 1$. Let $A_1 = \ell_B \cap \ell_C$, and let $a_2 = a^2$ (in other words, A_2 is the reflection of P across the diameter of ω through A). Define the points B_1, C_1, B_2, C_2 similarly.

We claim that $\overline{A_1A_2}$, $\overline{B_1B_2}$, $\overline{C_1C_2}$ concur at a point on Γ.

We begin by finding A_1. If we reflect the points $1+i$ and $1-i$ over $\overline{AB}$, then we get two points Z_1, Z_2 with

$$z_1 = a + b - ab(1-i) = a + b - ab + abi$$
$$z_2 = a + b - ab(1+i) = a + b - ab - abi.$$

Therefore,

$$z_1 - z_2 = 2abi$$
$$\overline{z_1}z_2 - \overline{z_2}z_1 = -2i\left(a + b + \frac{1}{a} + \frac{1}{b} - 2\right).$$

Now ℓ_C is the line $\overline{Z_1Z_2}$, so with the analogous equation ℓ_B we obtain (using the full formula in Theorem 6.17):

$$\begin{aligned}
a_1 &= \frac{-2i\left(a+b+\frac{1}{a}+\frac{1}{b}-2\right)(2aci)+2i\left(a+c+\frac{1}{a}+\frac{1}{c}-2\right)(2abi)}{\left(-\frac{2}{ab}i\right)(2aci)-\left(-\frac{2}{ac}i\right)(2abi)} \\
&= \frac{[c-b]\,a^2+\left[\frac{c}{b}-\frac{b}{c}-2c+2b\right]a+(c-b)}{\frac{c}{b}-\frac{b}{c}} \\
&= a+\frac{(c-b)\left[a^2-2a+1\right]}{(c-b)(c+b)/bc} \\
&= a+\frac{bc}{b+c}(a-1)^2.
\end{aligned}$$

Then the second intersection of $\overline{A_1A_2}$ with ω is given by

$$\begin{aligned}
\frac{a_1-a_2}{1-a_2\overline{a_1}} &= \frac{a+\frac{bc}{b+c}(a-1)^2-a^2}{1-a-a^2\cdot\frac{(1-1/a)^2}{b+c}} \\
&= \frac{a+\frac{bc}{b+c}(1-a)}{1-\frac{1}{b+c}(1-a)} \\
&= \frac{ab+bc+ca-abc}{a+b+c-1}.
\end{aligned}$$

Thus, the claim is proved.

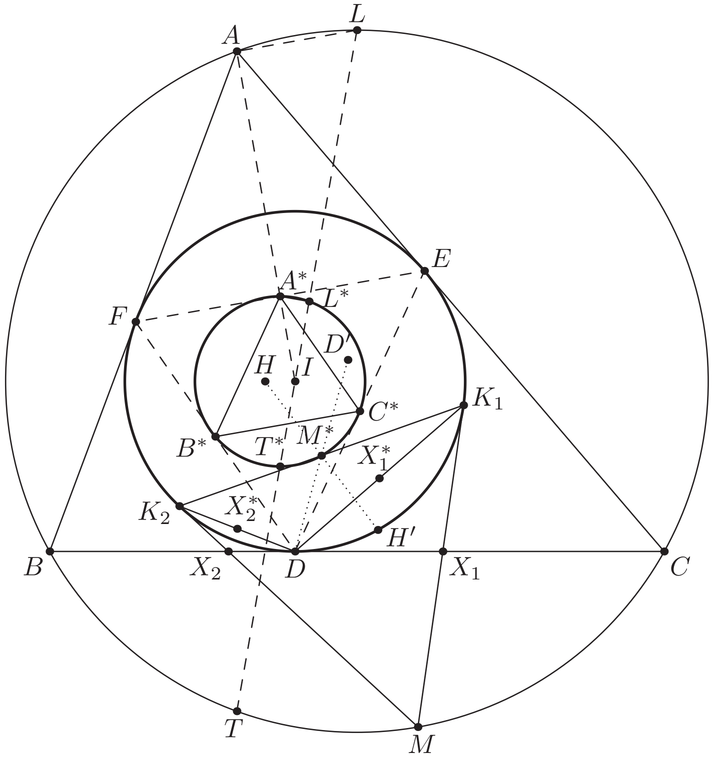
Finally, it suffices to show $\overline{A_1B_1} \parallel \overline{A_2B_2}$. Of course one can also do this with complex numbers, but it is easier to just use directed angle chasing[†] Let $\overline{BC}$ meet ℓ at K and $\overline{B_2C_2}$ meet ℓ at L. Evidently

$$\begin{aligned}
-\measuredangle B_2LP &= \measuredangle LPB_2+\measuredangle PB_2L \\
&= 2\measuredangle KPB+\measuredangle PB_2C_2 \\
&= 2\measuredangle KPB+2\measuredangle PBC \\
&= -2\measuredangle PKB \\
&= \measuredangle PKB_1
\end{aligned}$$

as required.

[†] One can also compute this more robustly using the notation $\measuredangle(\ell_1,\ell_2)$ to mean the directed angle $\measuredangle X_1OX_2$, where O is the intersection of lines ℓ_1 and ℓ_2 and X_1 and X_2 are any other points on ℓ_1, ℓ_2, respectively.

Solution 11.21



We know from Lemma 4.40 that the line TI passes through the midpoint of arc $\widehat{BC}$ containing A; call this point L.

Set DEF as the contact triangle of ABC. Let K_1 and K_2 be the contact points of the tangents from M (so that X_1 lies on $\overline{MK_1}$ and X_2 lies on $\overline{MK_2}$) and perform an inversion around the incircle. As usual we denote the inverse with a star. Now A^*, B^*, C^* are respectively the midpoints of $\overline{EF}$, $\overline{FD}$, $\overline{DE}$, and as usual $\Gamma^* = (A^*B^*C^*)$ is the nine-point circle of $\triangle DEF$.

Clearly M^* is an arbitrary point on Γ^*; moreover, it is the midpoint of $\overline{K_1K_2}$. Now let us determine the location of T^*. We see that L^* is some point also on Γ^*. Moreover,

$$\measuredangle IL^*A^* = -\measuredangle IAL = 90^\circ.$$

But because L, I, T are collinear it follows that L^*, I^*, T^* are collinear, whence

$$\measuredangle TL^*A^* = \measuredangle I^*L^*A^* = 90^\circ$$

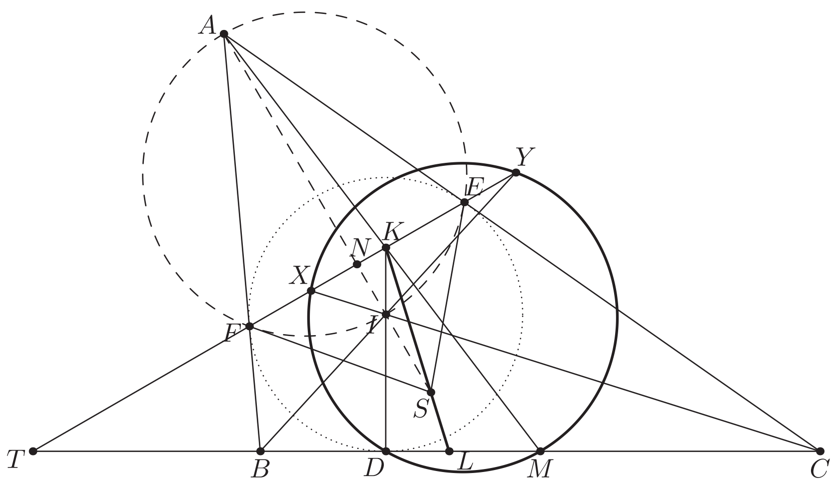
so T^* is the point diametrically opposite A^* on Γ^*. That means it is also the midpoint of $\overline{DH}$, where H is the orthocenter of triangle DEF.

It is now time to prove that M^*, X_1^*, X_2^*, T^* are concyclic. Dilating by a factor of 2 at D, it is equivalent to prove that D', K_1, K_2, and H are concyclic, where D' is the reflection of D over M^*. Reflecting around M^* it is equivalent to prove that D, K_2, K_1, and H' are concyclic.

But the circumcircle of D, K_2 and K_1 is just Γ^* itself. Moreover our usual homothety between the nine-point circle Γ^* and the incircle implies that H' lies on Γ^* as well. So D, K_2, K_1, H' are concyclic on Γ^*. Thus M, X_1, X_2, and T are concyclic, which is what we wanted to show.

Solution 11.22

Let D be the foot from I to $\overline{BC}$. Let X and Y denote the feet from B and C to $\overline{CI}$ and $\overline{BI}$. By Lemma 1.45, points X and Y lie on line EF. Let M be the midpoint of $\overline{BC}$, and ω the circumcircle of $DMXY$. By Lemma 9.27, the problem reduces to showing that T lies on the polar of S to ω.



Let $K = \overline{AM} \cap \overline{EF}$. By Lemma 4.17, points K, I, D are collinear. Let N be the midpoint of $\overline{EF}$, and set $L = \overline{KS} \cap \overline{BC}$. From

$$-1 = (A, I; N, S) \stackrel{K}{=} (T, L; M, D)$$

and

$$-1 = (T, D; B, C) \stackrel{I}{=} (T, K; Y, X)$$

we find that $T = \overline{MD} \cap \overline{YX}$ is the pole of $\overline{KL}$ with respect to ω, completing the proof.

APPENDIX D

List of Contests and Abbreviations

APMO Asian-Pacific Mathematical Olympiad. Started in 1989, the APMO is a regional competition for countries in the Asian Pacific region, as well as the United States and some other countries. The test consists of a single four-hour day with five problems.

BAMO Bay Area Mathematical Olympiad. The contest is taken by several hundred students in the Bay Area annually. The format is identical to that of the APMO.

Canada Canadian Mathematical Olympiad, abbreviated CMO.

CGMO The China Girls Mathematical Olympiad. The contest began in 2002, and consists of two days, each with four problems to be solved in four hours.

EGMO The European Girls' Mathematical Olympiad, a new contest inspired by the CGMO. The first EGMO was held in Cambridge in April 2012. Currently, the contest format matches the IMO. Countries send teams of up to four female students to compete at each event.

ELMO The ELMO is a contest held at MOP every year, produced by returning MOPpers and taken by first-time MOPpers. In particular, all problems are created, compiled, and selected by students.
The meaning of the acronym changes each year, originally standing for "Experimental Lincoln Math Olympiad" but soon taking such names as "Exceeding Luck-Based Math Olympiad", "Ex-experimental Math Olympiad", "$e^{\log}$ Math Olympiad", "End Letter Missing", "Entirely Legitimate (Junior) Math Olympiad", "Earn Lots of MOney", "Easy Little Math Olympiad", "Every Little Mistake $\Rightarrow 0$", "Everybody Lives at Most Once", and "English Language Master's Open".

ELMO Shortlist Like the IMO Shortlist, the ELMO Shortlist consists of problems proposed for the ELMO.

IMO The International Mathematical Olympiad, the supreme high school mathematics contest. Started in 1959, it is the oldest of the international science olympiads. The IMO draws in over 100 countries every July, and each country sends at most six students. On each of two days of the contest, contestants face three problem over 4.5 hours—problems are scored out of 7 points, so the maximum score is 42.

IMO Shortlist The IMO Shortlist, consisting of problems proposed for the IMO. About 30 problems are selected from all proposals (usually more than 100) to form the IMO shortlist. Team leaders from each country then vote a few days in advance on which problems from the shortlist will be selected to appear on the IMO. The IMO Shortlist of year N is not public until after the IMO of year $N + 1$, as many countries use shortlist problems in their national team selection tests.

JMO Short for USAJMO.

NIMO The National Internet Math Olympiad is an online contest written by a small group of students. The winter olympiad (from which the problems here are taken) is a one-hour exam for teams of up to four, and consists of eight problems.

OMO The Online Math Open. The Online Math Open is another online contest also administered completely by some of the top students in the USA. Teams of up to four students are given about a week to answer several short-answer problems, ranging from very easy to extremely difficult.

MOP Mathematical Olympiad Summer Program. MOP is the training camp for the USA team for the IMO; students are selected based on performance at the USA(J)MO. Until 2014, the camp was generally held in Lincoln, Nebraska during June for 3.5 weeks. Four-hour tests are given regularly at MOP. Several problems from this text are taken from such exams.

Sharygin The Russian Sharygin Geometry Olympiad is an international contest consisting solely of geometry problems. All problems in this book are taken from the Sharygin correspondence round, where students are given an extended period of time to submit solutions to several problems. Winners of the correspondence round are invited to Dubna, in Russia, for a final oral competition.

Shortlist See IMO Shortlist.

TST Abbreviation for Team Selection Test. Most countries use a TST as the final step in the selection of their team for the IMO.

USAJMO The USA Junior Mathematical Olympiad. It is an easier contest given at the same time as the USAMO for students in grades 10 and below. The format is identical to the USAMO.

USAMO USA Mathematical Olympiad. The USAMO is given to approximately 250 students each year, and used as part of the selection process for the USA team at the IMO, as well as for invitations to MOP. The format is identical to the IMO.

USA TST The Team Selection Test for the USA team. Up to 2011, the USA TST consisted of three days, each matching a day of the IMO. Since 2011 the TST has become more variable in its format, and is given only to the top eighteen students from the previous year's MOP.

USA TSTST The unfortunately-named "Team Selection Test for the Selection Team" is given at the end of MOP. It selects 18 students (the "selection team") to take further tests throughout the upcoming school year. The TSTST consists of two or three days, each matching the format of a day at the IMO.

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About the Author

Evan Chen is a past contest enthusiast hailing from Fremont, CA. In 2014 he was a winner of the USA Mathematical Olympiad, and earned a gold medal at that year's International Mathematical Olympiad. He is currently an undergraduate studying in Cambridge, Massachusetts, where he serves as problem czar for the Harvard-MIT Math Tournament.