The Use of Faces to Represent Points in k-Dimensional Space Graphically

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A novel method of representing multivariate data is presented. Each point in k-dimensional space, $k \leq 18$, is represented by a cartoon of a face whose features, such as length of nose and curvature of mouth, correspond to components of the point. Thus every multivariate observation is visualized as a computer-drawn face. This presentation makes it easy for the human mind to grasp many of the essential regularities and irregularities present in the data. Other graphical representations are described briefly.

1. INTRODUCTION

A graphical method of representing points in k-dimensional space is presented for consideration. The object is to represent multivariate data, subject to strong but possibly complex relationships, in such a way that an investigator can quickly comprehend relevant information and then apply appropriate statistical analysis. The method consists of representing a point in k-dimensional space by a drawing of a face whose characteristics are determined by the position of the point. A sample of points in k-dimensional space is represented by a collection of faces.

In the next section, two illustrations are sketched briefly. In one of these, where the investigator was interested in a cluster analysis, his task was merely to group together those faces which resembled each other. In the second, where the investigator was interested in detecting time points where a multivariate stochastic process changed character, he had to look at the sequence of faces corresponding to successive points in time to locate the places where the faces changed character.

Following sections discuss the potential advantage of this graphical method over that of looking at numerical data and consider some alternative approaches to and predecessors of this method. Some documentation which relates the data to the faces is contained in the Appendix.

2. ILLUSTRATIONS

We present two examples illustrating this representation.

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2.1 Fossil Data

Eight measurements were made on each of 88 nummulited specimens from the Eocene Yellow Limestone Formation of northwestern Jamaica. Two measurements thought to be age-dependent were discarded. One specimen (Number 34) was rejected because of a permutation in an early copy of the measurements for that specimen which cast doubt upon its accuracy. The data for Example 1 (see Table 1′ in the appendix) represent

Z_1 inner diameter of embryonic chamber (in microns)
Z_2 total number of whorls
Z_3 number of chambers in first whorl
Z_4 number of chambers in last whorl
Z_5 maximum height of chambers in first whorl (in microns)
Z_6 maximum height of chambers in last whorl (in microns)

The 87 faces corresponding to the 87 remaining specimens are presented in sequential order as indicated in Figure 1A. This order was selected *after* the data had been grouped into three clusters by an analytic technique [9].

The number at the bottom and left of each face is a randomly selected code number. Because the data were handled in two subgroups, these code numbers are repeated twice but half were marked with a cross. The sequence or i.d. numbers at the bottom right were added for publication. It is immediately obvious how these faces divide into three distinct clusters. This division is obvious partly because of the order in which the faces are presented. When copies were made, separated and mixed up without i.d. numbers and then given to people to cluster visually, these people selected the same clusters. On several occasions there were one or two discrepancies. The people, having the random code number but not the sequence number had no way of knowing except through the faces what grouping was expected.

A follow-up attempt to separate the large cluster of the first 40 faces (1–41 with 34 omitted) into subclusters seemed to be difficult, and results by various individuals were inconsistent with one another. Since the ranges of the variables in the first 40 specimens were smaller than for the 87 specimens, it seemed reasonable to magnify the effects of variation by renormalizing the data according to the ranges in the first 40 specimens. A new set of faces

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1A. FACES FOR 87 FOSSIL SPECIMENS OF EXAMPLE 1

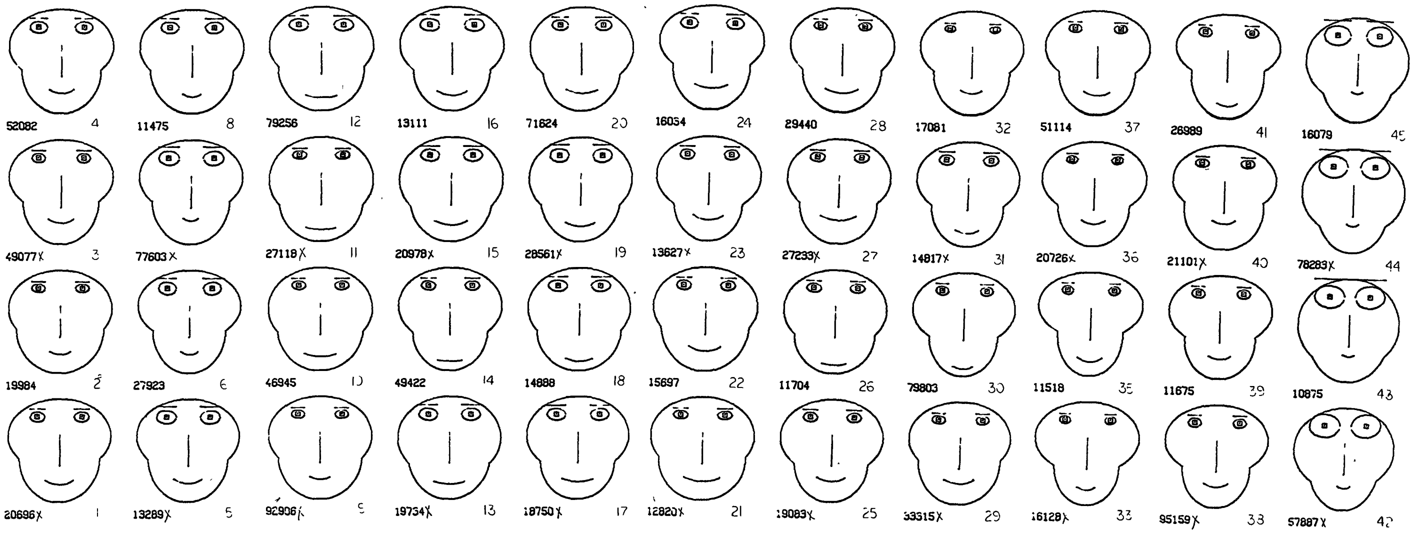
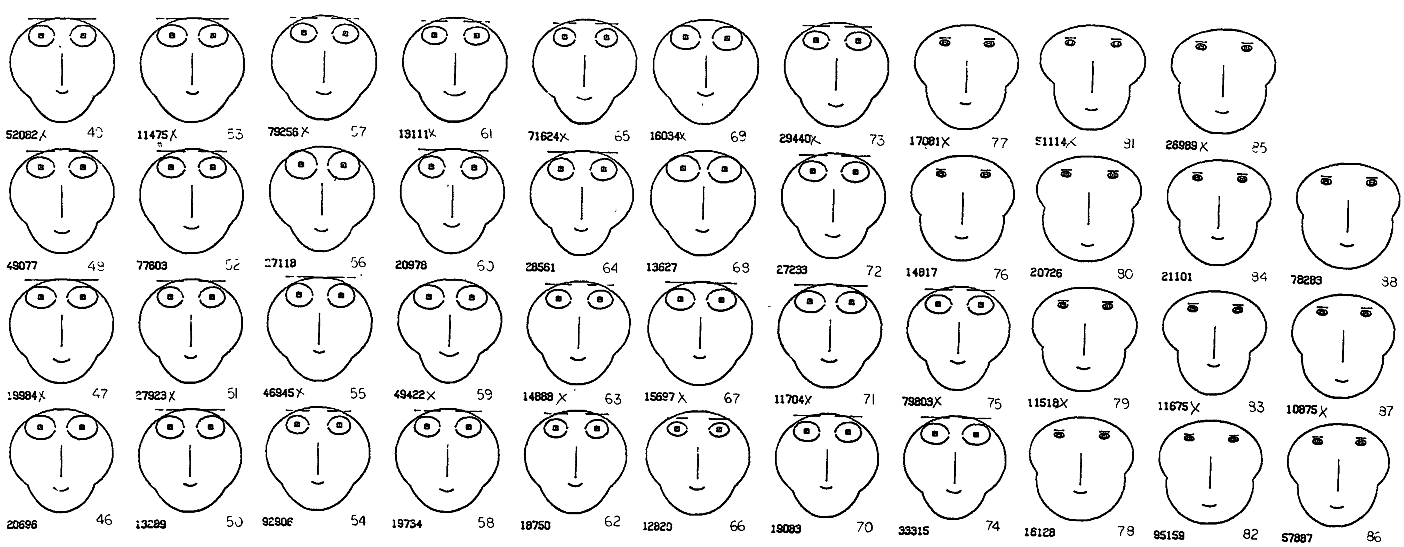


FIG. IA



was produced and is presented in Figure 1B. I clustered these visually. The groups were:

I: (1, 2, 3, 9, 22, 29)
II: (4, 5, 6, 7, 8)
III: (10, 11, 14, 23, 25, 26, 27)
IVa: (13, 15, 16, 17, 18, 19, 20)
IVb: (12, 24)
V: (21, 28, 30, 31, 37, 38, 39, 40, 41)
VI: (32, 33, 35, 36)

where IVb seemed to be similar to but slightly different from IVa. Professor Switzer also clustered them visually. He obtained:

Ia: (1, 2, 3, 9, 22, 29)
Ib: (4, 5, 12, 24)
Ic: (13, 15, 16, 17, 18, 19)
II: (6, 7, 8)
III: (10, 11, 14, 20, 21, 23, 25, 26, 27, 28, 31)
IV: (30, 32, 33, 35, 36, 37, 38, 40, 41)

which, though not in complete agreement with my groups, has substantial similarity.

As presented these lists do not completely do justice to the information obtained from the faces. They do not indicate which specimens are obviously members of a group and which were regarded as borderline. Each specimen was forced into some group, whereas one would ordinarily be inclined to separate out peculiar cases or outliers.

Finally, in connection with this example a graph of (Z_5, Z_6) is presented for these specimens since these variables seemed important in the set of 87 specimens. See Figure 1C.

2.2 Geological Data

Mineral analysis data from a 4,500-foot core drilled from a Colorado mountainside yielded 12 variables, presented in Table 2′ of the appendix. These represent assays of seven mineral contents by one method and repeated assays of five of these by a second method. These twelve variables were observed on each of 53 equally spaced specimens along the core and are presented in Table 2′ in the appendix.

The 53 faces obtained are shown in Figure 2 in the sequence as marked. They clearly indicate the sequence number where certain critical changes take place. One substantial change begins to take place after specimen 219, and those from 224 to 231 are quite distinct from the others. Another substantial change evolves from specimens 231 to 234. Particularly characteristic of the group from 224 to 231 are the tiny and high eyes, round face, broad smile, with mouth close to the relatively long nose. The group from 235 to 252 are characterized by a different constellation of special features, suggesting that a traditional linear analysis of this 12-dimensional time series may disguise some of the phenomena clearly observable.

3. POTENTIAL ADVANTAGES

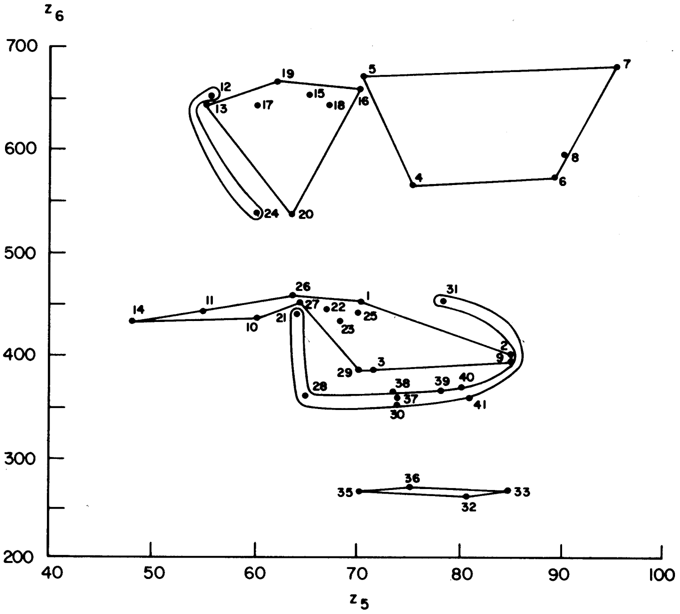
Graphical representations have many uses. These include (1) enhancing the user's ability to detect and comprehend important phenomena, (2) serving as a mnemonic device for remembering major conclusions, (3) communicating major conclusions to others, and (4) providing the facility for doing relatively accurate calculations informally. The representation by faces seems to have potential in the first two of these uses.

People grow up studying and reacting to faces all of the time. Small and barely measurable differences are easily detected and evoke emotional reactions from a long catalogue buried in the memory. Relatively large differences go unnoticed in circumstances where they are not important. This implies that the human mind subconsciously operates as a high-speed computer, filtering out insignificant visual phenomena and focusing on the potentially important. Particularly valuable is this flexibility in disregarding non-informative data and searching for useful information. It is this flexibility which is lacking in standard computer programs.

Moreover, this ability is great when applied to the study of faces. Experience with caricatures and cartoons would seem to indicate that the need for realistic faces on pictures is not great and that lack of realism is compensated for, at least in part, by the ability to caricaturize.

The ability to relate faces to emotional reactions seems to carry a mnemonic advantage. For example, in looking at the numerical data from the geological problem, major changes in individual variables are readily apparent. The author found that when studying these numerical data visually with no background in the scientific problem, many changes were observed but attention would be distracted quickly by other effects. After a substantial time, a confusion of reactions remained with little useful memory. Certain major characteristics of the faces are instantly observed and easily remembered in terms of emotions and appearance. Finer details and correlations become apparent after studying

IC. PLOT OF (Z_5,Z_6) FOR FIRST 40 OBSERVATIONS OF EXAMPLE 1 ON FOSSIL DATA ILLUSTRATING ONE OF GROUPINGS DERIVED FROM FACES OF FIGURE 1B



1B. FACES FOR FIRST 40 OBSERVATIONS OF EXAMPLE 1

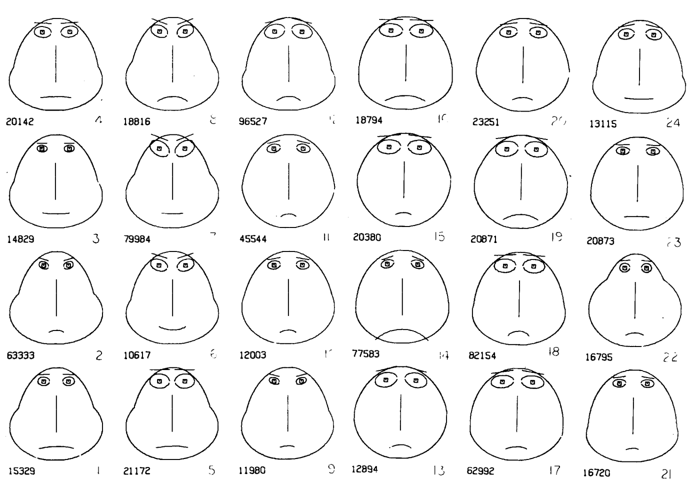
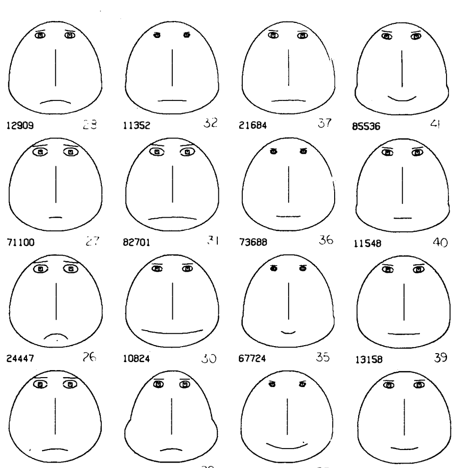


FIG. 1B



2. FACES FOR 53 GEOLOGICAL SPECIMENS OF EXAMPLE 2

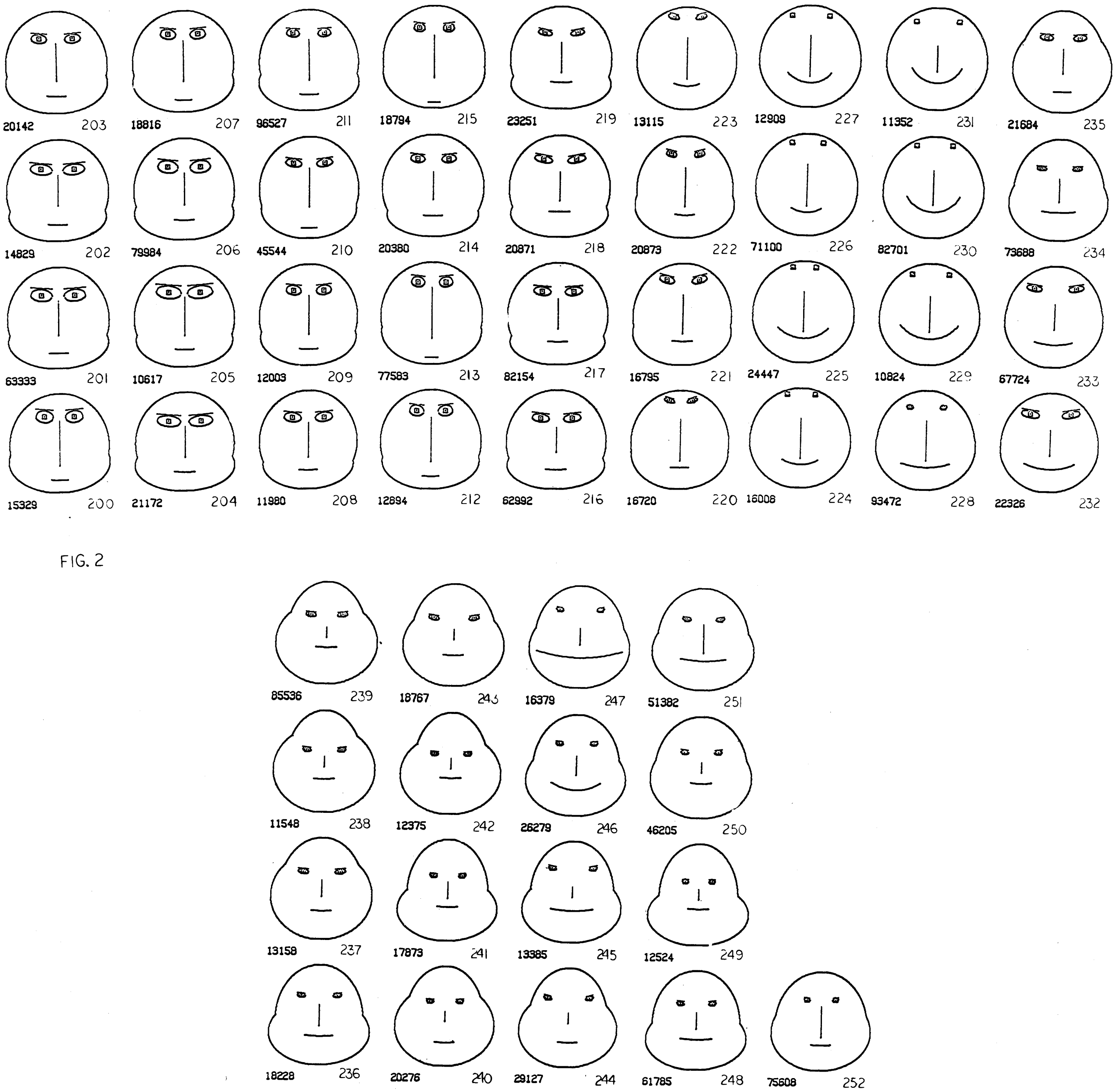


FIG. 2

the faces for a time. The awareness of these does not drive out of mind the original major impressions.

I would anticipate that the faces would have relatively little usefulness as a communication device. However, if results from a study of data were translated from the data to the faces, then the mnemonic advantages of the faces could conceivably make it desirable to use faces to communicate a relatively large assortment of results of varying degrees of importance.

Anyone who uses graph paper to analyze data is aware of how relatively refined estimates can be derived from rough drawings which are strategically arranged. I would anticipate that the faces do not lend themselves to any but the grossest kinds of estimates. The major advantage to be derived from using the faces should be in the heightened qualitative awareness of which numerical calculations are relevant.

4. ALTERNATIVE REPRESENTATIONS

One is led to ask two questions. First, if this simple idea is *so good*, why wasn't it thought of before? Second, what alternative representations are there for points in high-dimensional space?

Introspection would suggest that this idea must have been considered before in a simpler form. However, the effective application in this form would require a computer technology, which has only recently become available. Thus it is unlikely to have been used in the past in this form.

Recently a similar approach has been under study by Lindberg [4] who assembles a face from a collection of realistic features including the chin, hair, eyes, nose, lips, eyebrows, mustache (or glasses), and age lines into a composite realistic photograph. The features are used to represent measurements, ordinarily quantized into ten levels, which are matched to the ten "sizes" of the facial features.

Several more primitive attempts at representing points in k-dimensional space have come to my attention. Anderson [1] developed a method of using "glyphs," which are circles of fixed radius with rays of various lengths and directions extending from the boundary. The length of the ray represents the value of a variable. Pickett and White [5] used triangles which represent four variables (the three lengths of the sides and the orientation[1]). Both the glyphs and the triangles can raise the dimensionality by two by locating the center on a point in two-dimensional space. I have some memory of being told of a scheme to convert cardiograms or brain waves to sound in the hope that the human processing of sound would be more revealing than looking at graphs. This idea was explored by Speeth [8] in the use of seismograms to distinguish between earthquakes and nuclear explosions.

Several alternative representations have been considered. The most standard is the use of profiles. Here one represents a point in k-dimensional space by a series of k bars at heights corresponding to the values of the variables. It would seem desirable to standardize each variable so that the ranges either go from 0 to 1 or center about the mean. In some variations the bars are replaced by a polygonal line. This method was applied to the first 40 observations of Example 1, and, to my possibly prejudiced eyes, seemed to lack the penetrating power of the faces.

A much more effective variation of the profile method is one where a circle is drawn and along k equally spaced rays from the center, points are marked whose distance from the circumference is equal to standardized distances from the means of the k variables. These points are connected to form a polygon. The polygons resulting from this variation seem to be more readily translatable to human experience than the simpler profiles. They assume "meaningful" shapes, and the tendency to lean in certain directions has mnemonic force. This method apparently has some history of informal use in the medical profession and was applied in [6] to communicate the difference in profiles between the means of several groups. It was independently developed by Daetz [3].

[1] I have felt greatly indebted to Pickett for many conversations we had in which he emphasized how the human ability to process subconsciously large amounts of information of textures is fundamental to the ability to locomote and, indeed, to exist. After having developed the faces, I noticed his paper containing the triangle representation and realized that I had seen it before but had not paid special attention to it in the form presented.

A new technique of Andrews [2] consists of generating a Fourier Series of the form

$$f(t) = \frac{x_1}{\sqrt{2}} + x_2 \cos t + x_3 \sin t + x_3 \cos 2t + \cdots, \quad -\pi < t < \pi$$

where the x_i are the observed variables. This method has the interesting property that if x generates f, and y generates g, then

$$\int_{-\pi}^{\pi} [f(t) - g(t)]^2 dt = \pi \sum_{i=1}^{k} (x_i - y_i)^2,$$

suggesting that the method could be useful for expressing moderately refined calculations relevant to linear analysis. Andrews has applied this method using the principal components, in place of the original observations, for the x_i. In the normal multivariate model, these x_i would be independent, and the distances in the preceding expressions would be quite meaningful and unaltered by the transformation to principal components. A value of t which consistently and widely separates the $f(t)$ of two classes of observations provides an effective linear function for discriminating between the two classes.

It seems reasonable to conjecture that one may, in the spirit of the previously mentioned variation on profiles, achieve more suggestive curves by plotting $[f(t) + C, t]$ in polar coordinates.

5. SUMMARY

The use of the face representation provides a promising approach for a first look at multivariate data which is effective in revealing rather complex relations not always visible from simple correlations based on two-dimensional linear theories. It can be used to aid in cluster analysis, discrimination analysis, and to detect substantial changes in time series.

The study of faces does not seem to become more difficult as the number of variables increases. Generating faces with only the first seven and with the last five of the twelve variables of the data of Example 2 has indicated that the information content transmitted seems to become richer as the number of variables containing useful information increases. However, as one referee pointed out, if the useful information is in only a few of the variables, the presence of *noise* in many other variables may tend to diminish the ability to discern the useful information. At this point one can treat up to 18 variables,[2] but it would be relatively easy to increase that number by adding other features such as ears, hair, facial lines, and even possibly by taking pairs of faces.

This approach is an amusing reversal of a common one in artificial intelligence. Instead of using machines to discriminate between human faces by reducing them to numbers, we discriminate between numbers by using the

[2] We shall note in the appendix that the normalization of the width and length of the faces almost eliminates two of these variables.

1'. 6 MEASUREMENTS ON 87 NUMMULITED SPECIMENS FROM THE EOCENE YELLOW LIMESTONE FORMATION, JAMAICA

ID	z_1	z_2	z_3	z_4	z_5	z_6
1	160	51	10	28	70	450
2	155	52	8	27	85	400
3	141	49	11	25	72	380
4	130	50	10	26	75	560
5	161	50	10	27	70	665
6	135	50	12	27	88	570
7	165	50	11	23	95	675
8	150	50	9	29	90	580
9	148	48	8	26	85	390
10	150	45	7	31	60	435
11	120	40	6	33	55	440
12	120	51	8	32	56	650
13	100	42	8	30	55	640
14	100	44	9	35	48	430
15	150	40	7	29	65	650
16	90	46	9	30	70	655
17	75	42	8	28	60	640
18	120	47	7	35	67	645
19	200	43	9	30	62	660
20	120	41	8	28	63	530
21	105	50	7	27	64	435
22	210	52	9	26	67	440
23	90	40	10	25	68	430
24	110	52	11	25	60	530
25	100	43	9	25	70	440
26	90	44	7	36	63	454
27	70	45	8	23	64	450
28	100	48	9	27	65	355
29	130	52	9	25	70	380
30	90	45	11	37	74	350
31	80	46	10	32	78	450
32	95	49	10	25	82	260
33	70	44	12	30	85	262
35	95	51	15	31	70	270
36	100	46	11	24	76	270
37	95	48	10	27	74	355
38	85	47	12	25	73	360
39	70	48	11	26	78	365
40	80	54	10	21	80	370
41	85	55	13	33	81	355
42	200	34	10	24	98	1210
43	260	31	8	21	110	1220
44	195	30	9	20	105	1130
45	195	32	9	19	110	1010
46	220	33	10	24	95	1205
47	220	30	8	25	90	1210

ID	z_1	z_2	z_3	z_4	z_5	z_6
48	190	34	9	26	96	1070
49	285	30	11	19	100	990
50	300	30	9	20	102	1120
51	225	30	10	22	105	985
52	260	34	8	22	97	1090
53	280	30	8	20	112	1200
54	300	34	10	20	108	835
55	310	30	11	19	106	1055
56	290	31	12	26	94	1240
57	260	30	8	22	98	1015
58	290	33	9	25	100	1010
59	160	31	11	20	79	1170
60	240	35	11	20	88	990
61	195	31	8	21	81	975
62	290	34	10	19	94	860
63	210	35	9	22	96	950
64	180	30	11	22	97	990
65	205	29	11	23	90	805
66	215	34	8	21	100	700
67	270	31	8	20	111	1170
68	290	30	9	23	102	1350
69	320	32	10	19	87	1160
70	210	30	9	18	112	1010
71	210	30	8	21	95	1190
72	185	34	9	25	96	1055
73	200	32	8	26	98	980
74	170	29	9	20	95	1095
75	140	30	9	20	98	990
76	90	52	8	24	120	210
77	110	49	9	22	130	220
78	100	56	8	19	128	216
79	95	49	8	24	124	218
80	65	62	9	30	134	200
81	55	50	10	27	128	205
82	70	53	7	28	118	204
83	85	49	11	19	117	206
84	115	50	10	21	122	198
85	110	57	9	26	125	230
86	95	48	8	27	114	228
87	95	49	8	29	118	240
88	120	61	9	24	120	244
min	70	29	6	18	48	198
max	320	62	15	37	134	1350
min[1]	70	40	6	21	48	260
max[1]	210	55	15	37	95	675

[1] The second set of min and max refer to the ranges of the data over the first forty observations.

machine to do the brute labor of drawing faces and leaving the intelligence to the humans, who are still more flexible and clever.

One question frequently asked is whether some features are more informative than others. The individuals who worked on Example 1 felt that one only looked at eyes. The geologist believed that only the shapes of the head are relevant. In my opinion, the human will tend to concentrate on what is important in the data. However, this question requires serious study. At present an experiment is under way to determine whether permuting the variables has an effect on the ability of subjects to separate data from a mixture of the two normal multivariate distributions into the appropriate families.

Other related questions that arise are the following. Is the ability to use the faces too dependent on one's past experience? Will nonlinearities in reaction to cartoon figures lead to major misinterpretations?

In the meantime, there are a few obvious limitations which require care. When the eyes are very small, the position of the pupil becomes hard to detect. The zero point in the variable which controls the curvature of the mouth may have unusual significance and hence has been avoided in some studies. The corner points where the ellipses of the face meet disappear when the face is circular, leading to a loss of some information. These are minor points and can easily be avoided.

Two suggestions that may be useful for the investigator who feels that relatively few features are most discriminating and does not know which variables carry the useful information are the following: (1) try several permutations of the variables; (2) assign a sum of several variables to various features so that some variables appear in several features and some features are affected by several variables.

While the method looks promising, it still remains to be seen whether it can produce results not easily obtained by standard computations on the part of an investigator well versed in statistics and the field of application. One minor success was on the clustering of a randomly selected subset of Fisher's iris data which yielded poor results under the King stepwise clustering algorithm [7] However, nothing that I would regard as a convincing major success for this method has yet been obtained.

APPENDIX

A cartoon face is constructed using 18 variables $x_1, x_2, \cdots, x_{18}$ in appropriate ranges. The shape consists of two ellipses with horizontal and vertical major axes which are constructed as follows. Through a center point O a ray is drawn to a corner point P. The other corner point P' is taken so that OP and OP' are symmetric with respect to the vertical axis through O. The points U and L, respresenting the top and bottom of the face, are such that OU and OL are vertical and have equal length. The top of the face is the ellipse determined by U, P, P' and an eccentricity. The bottom of the face is similarly determined by L, P, P' and another eccentricity. The nose is a line segment centered about O. The mouth is an arc of a circle. The eyes and eyebrows are ellipses and line segments located symmetrically with respect to the vertical line through O. The pupils are both located the same horizontal distance from the center of the eyes. A normalization is then introduced to make both the width and length of the face equal to two inches. This normalization has the effect of reducing the effect of x_1 and x_3, which are described with the other parameters in Table 3′ which follows. Ordinarily an observation Z is normalized linearly so that the corresponding parameter $x = a + bZ$ covers a specified range. Except for the three eccentricities and the curvature of the mouth, these ranges are restricted to subsets of (0, 1). However, the eyebrow will tend to "cut" the eye if x_{16} is less than 0.2.

2′. DATA ON 12 VARIABLES REPRESENTING MINERAL CONTENTS FROM A 4500-FOOT CORE DRILLED FROM A COLORADO MOUNTAINSIDE

ID	Z_1	Z_2	Z_3	Z_4	Z_5	Z_6	Z_7	Z_8	Z_9	Z_{10}	Z_{11}	Z_{12}
200	320	105	057	050	001	001	001	060	020	250	210	370
201	280	150	040	050	001	001	001	060	040	210	130	420
202	260	165	033	050	001	001	001	060	010	250	090	440
203	305	110	044	040	001	001	001	050	050	260	140	250
204	290	160	035	035	001	001	001	050	020	210	060	510
205	275	130	047	035	001	001	001	050	020	230	090	570
206	280	155	035	035	001	001	001	080	020	270	170	400
207	300	115	050	060	001	001	001	120	010	280	190	300
208	250	130	041	030	005	001	001	070	030	250	110	330
209	285	120	047	040	001	001	001	070	010	240	170	280
210	280	105	047	070	001	001	001	060	020	370	070	300
211	300	135	050	040	001	001	001	120	060	250	160	200
212	280	110	056	050	001	001	001	150	010	280	270	280
213	305	080	065	080	005	001	001	130	010	300	260	260
214	230	175	029	035	001	001	001	270	030	250	140	240
215	325	060	052	090	001	001	001	160	010	280	260	170
216	270	170	025	040	001	001	001	160	010	290	070	330
217	250	185	031	025	001	001	001	120	001	260	080	330
218	260	185	030	015	001	001	001	270	080	480	010	330
219	270	185	032	010	005	001	001	180	040	450	020	220
220	325	045	053	005	020	001	001	600	080	660	020	250
221	315	090	047	005	020	001	001	410	200	600	060	260
222	335	100	047	010	040	001	001	360	080	590	110	170
223	310	010	049	005	080	018	001	640	240	630	060	190
224	410	001	049	001	075	032	001	760	440	800	001	001
225	360	001	048	001	080	055	001	770	260	770	010	010
226	310	015	051	001	105	036	001	660	380	640	001	010
227	420	005	049	001	095	056	001	620	520	680	001	001
228	415	020	049	005	025	036	001	370	220	340	001	001
229	420	005	041	001	070	060	001	630	510	580	001	001
230	450	005	040	001	090	070	001	690	570	630	001	001
231	395	001	025	015	100	071	001	580	530	560	001	010
232	380	010	027	025	035	039	001	350	320	400	001	270
233	430	010	025	030	030	025	001	340	340	360	001	200
234	410	075	022	010	005	015	001	170	170	170	001	060
235	520	055	024	040	005	001	001	210	190	190	001	180
236	385	135	018	010	005	008	001	140	200	260	001	020
237	535	065	010	020	001	001	001	110	230	270	001	070
238	550	095	001	010	001	001	001	050	230	270	001	030
239	510	100	001	001	001	001	001	190	150	230	001	110
240	510	095	001	040	001	001	001	140	100	150	001	040
241	385	180	010	001	001	001	001	050	050	300	001	050
242	505	125	001	001	001	001	001	001	200	130	001	030
243	470	090	001	020	001	001	001	160	300	380	001	060
244	465	110	001	035	001	001	001	260	440	500	001	060
245	400	140	001	015	001	023	001	330	400	390	001	040
246	415	105	015	025	040	032	001	220	190	270	001	010
247	435	075	010	015	001	069	001	370	360	500	001	010
248	370	145	010	010	005	012	040	130	080	330	001	030
249	380	210	001	001	001	001	020	070	001	050	001	030
250	430	065	001	005	020	001	075	130	070	300	001	020
251	420	080	030	001	005	026	001	050	100	350	001	050
252	425	060	035	005	001	001	030	100	010	340	001	010
min	250	001	001	001	001	001	001	001	001	050	001	001
max	520	210	065	090	105	071	075	770	570	800	270	570

3′. BRIEF DESCRIPTION PARAMETERS OF FACES AND RANGES USED IN EXAMPLES

		Subscript of Z and range used					
	Brief descriptive phrase	Fig. 1A		Fig. 1B		Fig. 2	
		Range	Data	Range	Data	Range	Data
x_1	radius to corner of face, $\lvert OP\rvert$	(0.3,0.8)	Z_1	0.9		0.8	
x_2	angle of OP to horizontal	(0.1,0.5)	Z_2	(0.2,0.8)	Z_1	(0.2,0.8)	Z_1
x_3	vertical size of face, $\lvert OU\rvert$	0.7		0.9		0.9	
x_4	eccentricity of upper face	(1.2,2.0)	Z_3	0.75		0.9	
x_5	eccentricity of lower face	1.0		(1.0,2.0)	Z_4	(1.0,2.0)	Z_2
x_6	length of nose	0.3		0.4		(0.1,0.5)	Z_3
x_7	vertical position of mouth	(0.2,0.8)	Z_4	0.5		(0.2,0.8)	Z_4
x_8	curvature of mouth	(0.5,5.0)	Z_5	(-4.0,4.0)	Z_3	(0.3,2.0)	Z_5
x_9	width of mouth	0.5		(0.2,0.8)	Z_4	(0.2,0.9)	Z_6
x_{10}	vertical position of eyes	0.5		0.5		(0.1,0.8)	Z_8
x_{11}	separation of eyes	0.5		0.5		(0.2,0.8)	Z_9
x_{12}	slant of eyes	0.5		(0.2,0.8)	Z_5	(0.5,0.8)	Z_{10}
x_{13}	eccentricity of eyes	0.6		0.6		(0.4,0.8)	Z_{11}
x_{14}	size of eyes	(0.2,0.8)	Z_6	(0.2,0.8)	Z_6	(0.2,0.8)	Z_{12}
x_{15}	position of pupils	0.5		0.5		0.5	
x_{16}	vertical position of eyebrows	0.5		0.5		(0.3,0.6)	Z_7
x_{17}	slant of eyebrows	0.5		0.5		0.5	
x_{18}	size of eyebrows	0.5		0.5		0.5	

At this time the cost of drawing these faces is about 20 to 25 cents per face on the IBM 360-67 at Stanford University using the Calcomp Plotter. Most of this cost is in the computing, and I believe that it should be possible to reduce it considerably.

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